

REPRESENTATIONS WITH SMALL K TYPES I

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ABSTRACT. Let $\mathfrak{g}_{\mathbb{R}}$ be a split real, simple Lie algebra with complexification \mathfrak{g} . Let $G_{\mathbb{C}}$ be the connected, simply connected Lie group with Lie algebra \mathfrak{g} , $G_{\mathbb{R}}$ the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{R}}$, and G a covering group of $G_{\mathbb{R}}$ with a maximal compact subgroup K . A complete classification of "small" K types is derived via Clifford algebras, and an analog, P^{ξ} , of Kostant's P^{γ} matrix is defined for a K type ξ of principal series admitting a small K type. For the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 , a product formula for the determinant of P^{ξ} over the rank one subgroups corresponding to the positive roots is proved. As a result, cyclicity of a small K type of principal series in the closed Langlands chamber is proved, which implies irreducibility of unitary principal series admitting a small K type.

1. INTRODUCTION

Let G be a real reductive Lie group with a maximal compact subgroup K and let $\mathfrak{g}_{\mathbb{R}} = \text{Lie}(G)$ and $\mathfrak{k}_{\mathbb{R}} = \text{Lie}(K)$. We drop the subscripts to denote the complexifications of the Lie algebras. In [Kos], Kostant studies (\mathfrak{g}, K) modules X that admit a trivial K type via the determinant of his P^{γ} matrix defined for each K type γ that occurs in X . Kostant determines this determinant by proving a product formula over the rank one subgroups corresponding to the reduced restricted roots and calculating the determinant in the rank one cases. He proves several implications of the determinant formula; in particular, the cyclicity of the trivial K type of the spherical principal series in the closed Langlands chamber, the irreducibility of the unitary spherical principal series, and the realization of X as a quotient of the spherical principal series in the closed Langlands chamber. In addition to the implications derived by Kostant, the product formula along with the realization of the intertwining operators between the spherical principal series representations of G as a ratio of the P^{γ} matrices derived by Johnson and Wallach [JW] give a method to compute the shift factors of Vogan

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and Wallach's generalization of Leslie Cohn's determinant formula for the restriction of the intertwining operator to a K isotypic component given in terms of ratios of classical gamma functions [VW].

We derive analogous results to Kostant's for (\mathfrak{g}, K) modules V that admit a "small" K type τ (see definition below) via the determinant of the generalization of Kostant's P^γ matrix, P^ξ , for a K type ξ that occurs in V for the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 . Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of a reductive Lie algebra, let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} and let 0M be the centralizer of \mathfrak{a} in K . In 11.A.2 of [RRG II], Wallach defines a small K type to be an irreducible representation (τ, V_τ) of K such that $\sigma = V_\tau|_{{}^0M}$ is irreducible, and gives examples for all real forms of all simple Lie types. Let $I_{P,\sigma,\nu}$ be the usual K finite principal series for a minimal parabolic subgroup P , σ , and $\nu \in \mathfrak{a}^*$. Let $Y^{\tau,\nu} = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^K} V_{\tau,\nu}$ be the (\mathfrak{g}, K) module where \mathfrak{g} acts by left translation, K acts by $\text{Ad} \otimes \tau$, and $U(\mathfrak{g})^K$ acts on $V_{\tau,\nu}$ as it does on $I_{P,\sigma,\nu}(\tau)$. Wallach also proves that $Y^{\tau,\nu}$ is isomorphic with $\text{Ind}_{{}^0M}^K(\sigma)$ as K modules. We also note that $I_{P,\sigma,\nu} \cong Y^{\tau,\nu}$ as K modules and that the spherical principal series is a special case as the trivial K type is a small K type.

Let now $\mathfrak{g}_\mathbb{R}$ be a split real, simple Lie algebra. Let $G_\mathbb{C}$ be the connected, simply connected Lie group with Lie algebra \mathfrak{g} , $G_\mathbb{R}$ the connected subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_\mathbb{R}$, and G a covering group of $G_\mathbb{R}$ with covering homomorphism p and a maximal compact subgroup K . The first main result of this article is the following classification of small K types. $Z = \ker(p)$. If K is a product of two groups, denote by p_1 and p_2 the projection onto the first factor and the second factor respectively. Denote by s the spin representation of $\text{Spin}(n)$ for n odd, and either of the two half spin representations of $\text{Spin}(n)$ for n even.

Theorem 1 (Classification of Small K types)

Type	K	Z	τ
A_n ($n \geq 2$)	$\text{Spin}(n+1)$	$\mathbb{Z}/2\mathbb{Z}$	s
B_n ($n \geq 3$)	$\text{Spin}(n+1) \times \text{Spin}(n)$	$\mathbb{Z}/2\mathbb{Z}$	$s \circ p_1$ or $s \circ p_2$ for n odd, $s \circ p_2$ for n even
C_n	$SO(2) \times SU(n)$	$\mathbb{Z}/m\mathbb{Z}$	$\{e^{\frac{\pi i k}{m}} \mid (k, m) = 1\}$
D_n ($n \geq 3$)	$\text{Spin}(n) \times \text{Spin}(n)$	$\mathbb{Z}/2\mathbb{Z}$	$s \circ p_1$ or $s \circ p_2$
E_6	$Sp(4)$	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{C}^8
E_7	$SU(8)$	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{C}^8 or $(\mathbb{C}^8)^*$
E_8	$\text{Spin}(16)$	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{C}^{16}
F_4	$Sp(3) \times SU(2)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{C}^2 \circ p_2$
G_2	$SU(2) \times SU(2)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{C}^2 \circ p_1$ or $\mathbb{C}^2 \circ p_2$

Based on the results above, for a K type ξ that occurs in the principal series $I_{P,\sigma,\nu}$, a generalization of Kostant's P^γ matrix, P^ξ , is defined (cf. Definition 3.1). The definition of P^ξ depends on the choice of a small K type if $I_{P,\sigma,\nu}$ admits more than one (cf. section 3). From this point on, the exposition is for the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 , which we denote by G from this point on. The second main result is a product formula for the determinant of P^ξ over the rank one subgroups corresponding to the positive roots. We note that for G , $I_{P,\sigma,\nu}$ admits only one small K type (cf. Theorem 1 and proof of Lemma 4.11). Let ϕ be a positive root of \mathfrak{g} , and let G_ϕ be the corresponding rank one subgroup. G_ϕ has its semisimple part the group generated by the metaplectic group $Mp_2(\mathbb{R})$ and 0M . Let K_ϕ be the maximal compact subgroup of G_ϕ generated by a torus and 0M . Let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}$ where V_{τ_j} is an irreducible K_ϕ submodule of V_ξ such that $V_{\tau_j} \cong V_\tau$ as 0M modules for all $j = 1, \dots, n(\xi)$ (cf. Theorem 4.6 and Lemma 4.11). Let $p_\phi = p_{\tau_1}^\phi \dots p_{\tau_{n(\xi)}}^\phi$ where $p_{\tau_j}^\phi$ is the determinant of P^{τ_j} matrix of the rank one case of G_ϕ with K_ϕ type τ_j . Denote by $p_{(\phi)} = T_{\rho_\phi - \rho}(p_\phi)$ where $T_{\rho_\phi - \rho}$ is translation by $\rho_\phi - \rho$. The following is a product formula for p_ξ , the determinant of P^ξ , over the rank one subgroups corresponding to the positive roots for the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 .

Theorem 2 There exists a nonzero scalar c such that

$$p_\xi(\nu) = c \prod_{\phi \in \Phi^+} p_{(\phi)}(\nu)$$

For the proof of the product formula we follow Kostant's strategy by proving that the rank one factors divide and proving that the degree is correct. The divisibility argument is similar to Kostant's; however, our approach to the degree is different. For this, we make a comparison of certain weight vectors of the torus corresponding to each positive root (Theorem 5.4).

The product formula has several implications. First, cyclicity of the small K type $I_{P,\sigma,\nu}(\tau)$ in the closed Langlands chamber (Theorem 6.4) is proved as a consequence of the nonvanishing determinant of $P^\xi(\nu)$ in the closed Langlands chamber for all $\xi \in \hat{K}$ that occur in $I_{P,\sigma,\nu}$. Second, irreducibility of the unitary principal series $I_{P,\sigma,\nu}$ ($\operatorname{Re} \nu = 0$) (Theorem 6.5) is proved as a consequence of the cyclicity of $I_{P,\sigma,\nu}(\tau)$. In addition, we derive the third main result, which is the following classification of irreducible (\mathfrak{g}, K) modules that admit a small K type, whose consequence is nonexistence of discrete series representation of G that admit a small K type (Corollary 6.7).

Theorem 3 Let G be the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 . For ν in the closed Langlands chamber, $I_{P,\sigma,\nu} \cong Y^{\tau,\nu} = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^K} V_{\tau,\nu}$ as (\mathfrak{g}, K) modules. If V is an irreducible (\mathfrak{g}, K) module containing τ , V is (\mathfrak{g}, K) isomorphic with the unique irreducible quotient of $Y^{\tau,\nu}$ (hence $I_{P,\sigma,\nu}$) for some ν in the closed Langlands chamber.

The applications of the product formula are the following. By the explicit computation of the determinant of the P^ξ matrix in the rank one case (cf. 6.2) and the realization of the intertwining operator between the genuine principal series representations of G as a ratio of the P^ξ matrices (Theorem 6.2) similarly as in the spherical case, we derive a method to find the unknown shift factors of the determinant of the intertwining operators given in terms of ratios of the classical gamma functions derived by Vogan and Wallach [VW]. We derive explicit formulas for $\widetilde{SL}(n, \mathbb{R})$ (cf. 6.2).

We now discuss the layout of the article. In section 2, we derive a complete classification of small K types. In section 3, we define for a K type ξ that occurs in the principal series representation admitting a small K type τ a matrix P^ξ (cf. Definition 3.1) that agrees with Kostant's P^γ matrix [Kos] if τ is the trivial K type. In section 4, we derive structural results for K modules that occur in \mathcal{H} or $\mathcal{H} \otimes V_\tau$ where \mathcal{H} is the space of harmonics on \mathfrak{p} . In section 5, we prove a product formula for the determinant of P^ξ over the rank one subgroups of G corresponding to the positive roots. In section 6, we derive consequences of the product formula for the determinant of P^ξ .

Note. For the connected, simply connected, split real form of type G_2 , there is an analogous proof of the product formula for p_ξ defined via the small K type $\mathbb{C}^2 \circ p_1$ where $p_1(K)$ is the long $SU(2)$. We are planning a sequel to this article containing the result for type G_2 and the rest of the connected, simply connected split real form of doubly laced, simple Lie types for which the analogous result is true.

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2. CLASSIFICATION OF SMALL K TYPES

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} , and let $\mathfrak{g}_{\mathbb{R}}$ be the split real form of \mathfrak{g} . Let $G_{\mathbb{C}}$ be the connected, simply connected Lie group with Lie

algebra \mathfrak{g} , and let $G_{\mathbb{R}}$ be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Let G be a covering group of $G_{\mathbb{R}}$ nonisomorphic to $G_{\mathbb{R}}$ with covering homomorphism $p : G \rightarrow G_{\mathbb{R}}$. Let K be a maximal compact subgroup of G defined as the subgroup of the fixed points of a Cartan involution θ , and let U be a compact form of $G_{\mathbb{C}}$ such that $G_{\mathbb{R}} \cap U = K_{\mathbb{R}} = p(K)$.

In this section, we derive a complete classification of small K types (see the definition below). We first derive a complete classification of small K types for G of type A_n via Clifford Algebras. For other types, we introduce appropriate embeddings of $\widetilde{SL(n, \mathbb{R})}$ or the metlinear group $GL(n, \mathbb{R})$ into G to derive a complete classification of small K types via the result for type A_n .

We also denote the differential of the above Cartan involution by θ . Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ be the Cartan decomposition and $\mathfrak{a}_{\mathbb{R}}$ be a maximal abelian subspace of $\mathfrak{p}_{\mathbb{R}}$. Let ${}^0M = {}^0M_G = Z_K(\mathfrak{a}_{\mathbb{R}})$ and ${}^0M_{G_{\mathbb{R}}} = Z_{K_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$.

Definition 2.1. (Wallach) An irreducible representation (τ, V_{τ}) of K is small if $\tau|_{{}^0M}$ is irreducible.

Let Z be the kernel of p . $Z \subset K$. Let χ be a genuine character of Z , i.e. χ is injective. Theorem 1 in the introduction gives a complete classification of small K types (τ, V_{τ}) such that $\tau|_Z = \chi Id$. We now prove Theorem 1 in the introduction.

Proof. (of Theorem 1)

The K types listed are small (11.A [RRG II]). We prove the K types give a complete list of small K types.

As $G_{\mathbb{R}}$ is split and has a complexification $G_{\mathbb{C}}$, ${}^0M_{G_{\mathbb{R}}} \cong (\mathbb{Z}/2\mathbb{Z})^n$ where each $\mathbb{Z}/2\mathbb{Z}$ is from each and every simple root of $\mathfrak{g}_{\mathbb{R}}$ and n is the rank of $\mathfrak{g}_{\mathbb{R}}$.

For type C_n , as $Z \subset SO(2)$, ${}^0M = p^{-1}({}^0M_{G_{\mathbb{R}}}) = p^{-1}((\mathbb{Z}/2\mathbb{Z})^n)$ is abelian. Therefore, small K types are necessarily 1 dimensional, and the K types in the table exhaust the list.

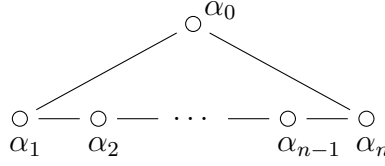
For the classification of small K types for type A_n ($n \geq 2$), we introduce Clifford Algebras. Let $V = \mathbb{R}^{n+1}$, $(\ , \)$ the standard inner product on V . Let $T(V)$ be the tensor algebra on V , and let I be the ideal of $T(V)$ generated by the elements $x \otimes x + (x, x)1$. Set $Cliff(V) = T(V)/I$. Let $\{e_1, \dots, e_{n+1}\}$ be the standard basis of V . $Cliff(V)$ is an algebra over \mathbb{R} generated by e_1, \dots, e_{n+1} with relations $e_j^2 = -1$ and $e_i e_j = -e_j e_i$ if $i \neq j$. Let $S = \{v \in \widetilde{\mathbb{R}^{n+1}} | (v, v) = 1\}$. The maximal compact subgroup $Spin(n+1)$ of $SL(n+1, \mathbb{R})$ is a subgroup of $Cliff(V)$ of products of even elements of S . $Spin(n+1)$ is a two fold

covering group of $SO(n+1)$, and ${}^0M_{\widetilde{SL(n+1, \mathbb{R})}}$ is generated by $\{e_i e_{i+1} \mid i = 1, \dots, n\}$ (11.A.2.6 [RRG II]).

$\mathbb{C}[{}^0M_{\widetilde{SL(n+1, \mathbb{R})}}]/\langle \eta + 1 \rangle$ is isomorphic to the Clifford Algebra on \mathbb{C}^n where η the nontrivial element of $Z = \mathbb{Z}/2\mathbb{Z}$. $Cliff(\mathbb{C}^n)$ is isomorphic to the simple matrix algebra $M_{2^{\frac{n}{2}}}(\mathbb{C})$ for n even and a direct sum of two simple matrix algebras $M_{2^{\frac{n-1}{2}}}(\mathbb{C}) \oplus M_{2^{\frac{n-1}{2}}}(\mathbb{C})$ for n odd (Proposition 6.1.5 & 6.1.6 [GW]). Hence a small K type must be of dimension $2^{\frac{n}{2}}$ for n even and $2^{\frac{n-1}{2}}$ for n odd. Weyl dimension formula implies the K types in the table are the only ones with appropriate dimensions.

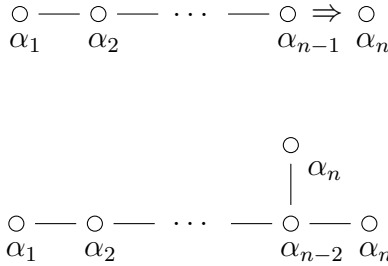
For types other than A_n , and C_n , we make use of the above analysis of type A_n . We first realize certain embedding $i : S_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}} = G/Z$ using Dynkin diagrams and extended Dynkin diagrams (Chapter 6 [Bou]).

- A_n : $S_{\mathbb{R}} \cong GL(n, \mathbb{R})$.



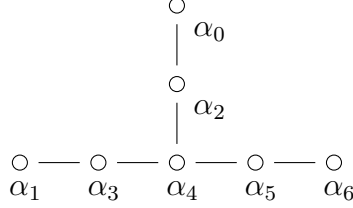
Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, \dots, \alpha_{n-1}$. The identity component of L is isomorphic with $SL(n, \mathbb{R})$. The embedded subgroup isomorphic to $GL(n, \mathbb{R})$ is generated by the $SL(n, \mathbb{R})$, $\mathbb{Z}/2\mathbb{Z} \subset {}^0M_{G_{\mathbb{R}}}$ from the node α_n , and $\mathbb{R}_{>0}$ from the node α_0 .

- B_n and D_n : $S_{\mathbb{R}} \cong SL(n, \mathbb{R})$.



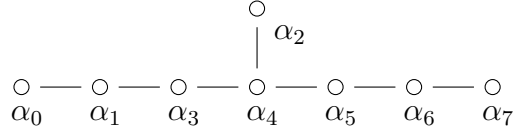
Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, \dots, \alpha_{n-1}$. The identity component of L is isomorphic with $SL(n, \mathbb{R})$.

- E_6 : $S_{\mathbb{R}} \cong GL(6, \mathbb{R})$.



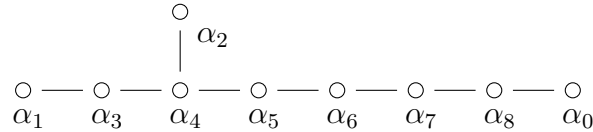
Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. The identity component of L is isomorphic with $SL(6, \mathbb{R})$. The embedded subgroup isomorphic to $GL(6, \mathbb{R})$ is generated by the $SL(6, \mathbb{R})$, $\mathbb{Z}/2\mathbb{Z} \subset {}^0M_{G_{\mathbb{R}}}$ from the node α_2 , and $\mathbb{R}_{>0}$ from the node α_0 .

- E_7 : $S_{\mathbb{R}} \cong SL(8, \mathbb{R})$.



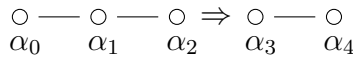
Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$. The identity component of L is isomorphic with $SL(8, \mathbb{R})$.

- E_8 : $S_{\mathbb{R}} \cong SL(9, \mathbb{R})$.



Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0$. The identity component of L is isomorphic with $SL(9, \mathbb{R})$.

- F_4 : $S_{\mathbb{R}} \cong Spin(5, 4)_0$.



Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. The identity component of L is isomorphic with $Spin(5, 4)_0$.

- G_2 : $S_{\mathbb{R}} \cong SL(3, \mathbb{R})$.

$$\begin{array}{ccc} \circ & \Leftarrow & \circ \text{ --- } \circ \\ \alpha_1 & & \alpha_2 \quad \alpha_0 \end{array}$$

Let P be the parabolic subgroup with Levi factor L where the simple roots of $Lie(L)$ are α_2, α_0 . The identity component of L is isomorphic with $SL(3, \mathbb{R})$.

The embedding $i : S_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$ lifts to the embedding $\tilde{i} : S \hookrightarrow G$ where S is a subgroup of G isomorphic to $p^{-1}(i(S_{\mathbb{R}}))$. For types B_n and D_n , ${}^0M \cong {}^0M_{\widetilde{SL(n, \mathbb{R})}} \times \mathbb{Z}/2\mathbb{Z}$ where $\widetilde{SL(n, \mathbb{R})} \cong S$ and the $\mathbb{Z}/2\mathbb{Z}$ can be either $(\pm 1, 1)$ or $(1, \pm 1)$ where ± 1 is the kernel of the covering homomorphism $p : Spin(m) \rightarrow SO(m)$ for $m = n$ or $n + 1$ (11.A.2.8 [RRG II]). For the rest of the types, $i({}^0M_{S_{\mathbb{R}}}) = {}^0M_{G_{\mathbb{R}}}$ as ${}^0M_{G_{\mathbb{R}}} \cong (\mathbb{Z}/2\mathbb{Z})^n$ where each $\mathbb{Z}/2\mathbb{Z}$ is from each and every simple root of $\mathfrak{g}_{\mathbb{R}}$ and n is the rank of $\mathfrak{g}_{\mathbb{R}}$. As ${}^0M_G = p^{-1}({}^0M_{G_{\mathbb{R}}}) = p^{-1}(i({}^0M_{S_{\mathbb{R}}})) = \tilde{i}({}^0M_S)$, we derive the following.

G	S	0M_G isomorphic to
A_n	$\widetilde{GL(n, \mathbb{R})}$	${}^0M_{\widetilde{GL(n, \mathbb{R})}}$
E_6	$\widetilde{GL(6, \mathbb{R})}$	${}^0M_{\widetilde{GL(6, \mathbb{R})}}$
E_7	$\widetilde{SL(8, \mathbb{R})}$	${}^0M_{\widetilde{SL(8, \mathbb{R})}}$
E_8	$\widetilde{SL(9, \mathbb{R})}$	${}^0M_{\widetilde{SL(9, \mathbb{R})}}$
F_4	$\widetilde{Spin(5, 4)}_0$	${}^0M_{\widetilde{Spin(5, 4)}_0} \cong {}^0M_{\widetilde{SL(4, \mathbb{R})}} \times \mathbb{Z}/2\mathbb{Z}$
G_2	$\widetilde{SL(3, \mathbb{R})}$	${}^0M_{\widetilde{SL(3, \mathbb{R})}}$

We find necessary dimensions of small K types using the discussion of type A_n . Weyl dimension formula implies the K types in the table exhaust the list of all small K types. \square

Consider now the metaleinear group $\widetilde{GL(n, \mathbb{R})}$ for $n \geq 3$ with a maximal compact subgroup $K = Pin(n)$. We give the following classification of small K types for use in the proof of the product formula for p_{ξ} . Let s be the pin representation or either of the two pin representations of $Pin(n)$ depending on the parity of n .

Theorem 2.1. *The small K type for $\widetilde{GL(n, \mathbb{R})}$ ($n \geq 3$) is s .*

Proof. As ${}^0M_{\widetilde{GL(n, \mathbb{R})}} \cong {}^0M_{\widetilde{SL(n+1, \mathbb{R})}}$ (cf. proof of Theorem 1), the analysis for the type A_n gives the necessary dimensions of a small K type τ and proves that s is the only small K type. \square

3. P^ξ MATRIX

Let G be as in section 2. In this section, we define a matrix, P^ξ , with entries in $U(\mathfrak{a})$ the universal enveloping algebra in $\mathfrak{a} = \mathfrak{a}_\mathbb{R} \otimes \mathbb{C}$ where ξ is a K type that occurs in the principal series representation of G that admits a small K type τ . If τ is the trivial K type, the definition of P^ξ matrix agrees with Kostant's P^γ matrix [Kos].

Let $P = {}^0MAN$ be a parabolic subgroup of G with a given Langlands decomposition. Let (σ, H_σ) be an irreducible Hilbert representation of 0M that is unitary when restricted to $K \cap {}^0M$. Let ${}^\infty H^{P,\sigma,\nu}$ be the space of all smooth functions $f : G \rightarrow H_\sigma$ such that $f(mang) = \sigma(m)a^\nu f(g)$ for $m \in {}^0M$, $a \in A$, $n \in N$, and $g \in G$ where $\nu \in \text{Lie}(A)_\mathbb{C}^*$. Define for $f, g \in {}^\infty H^{P,\sigma,\nu}$ the inner product $\langle f, g \rangle = \int_K \langle f(k), g(k) \rangle_\sigma dk$. Let $H^{P,\sigma,\nu}$ be the Hilbert space completion of ${}^\infty H^{P,\sigma,\nu}$. The right regular action $\pi_{P,\sigma,\nu}(g)f(x) = f(xg)$ gives a Hilbert Representation $(\pi_{P,\sigma,\nu}, H^{P,\sigma,\nu})$ of G called the principal series representation. If $X \in \mathfrak{g}_\mathbb{R}$, $X.f(g) = \frac{d}{dt}|_{t=0} f(g \cdot \exp(tX))$ defines a natural action of $\mathfrak{g}_\mathbb{R}$ on $H^{P,\sigma,\nu}$ induced from $\pi_{P,\sigma,\nu}$. We also denote the action of $\mathfrak{g}_\mathbb{R}$ and hence the action of \mathfrak{g} and its universal enveloping algebra $U(\mathfrak{g})$ by $\pi_{P,\sigma,\nu}$. For $\gamma \in \hat{K}$, let $H^{P,\sigma,\nu}(\gamma)$ be the sum of all K invariant, finite dimensional subspaces of $H^{P,\sigma,\nu}$ that are in the class of γ . Let $I_{P,\sigma,\nu}$ be the algebraic direct sum $\bigoplus_{\gamma \in \hat{K}} H^{P,\sigma,\nu}(\gamma) \cap {}^\infty H^{P,\sigma,\nu}$, the underlying (\mathfrak{g}, K) module of the principal series representation $(\pi_{P,\sigma,\nu}, H^{P,\sigma,\nu})$.

From this point on, let P be a minimal parabolic subgroup of G and let $\sigma = \tau|_{{}^0M}$ for a small K type τ . We drop the subscript \mathbb{R} to denote the complexifications of subspaces of $\mathfrak{g}_\mathbb{R}$ introduced in the last section. Let $Y^{\tau,\nu}$ be the (\mathfrak{g}, K) module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^K U(\mathfrak{k})} V_{\tau,\nu}$ with the action of \mathfrak{g} by left translation, the action of K by $Ad \otimes \tau$, and the action of $U(\mathfrak{g})^K$ and $U(\mathfrak{k})$ on $V_{\tau,\nu} = V_\tau \subset I_{P,\sigma,\nu}$ as a differential operator $\pi_{P,\sigma,\nu}$. $Y_{\tau,\nu} \cong I_{P,\sigma,\nu}$ as K modules (11.3.6 [RRG II]). Let \mathcal{H} be the space of harmonics on \mathfrak{p} , $\mathcal{J} = S(\mathfrak{p})^K$ the subspace of K invariants of the space of symmetric polynomials on \mathfrak{p} . Let $\text{symm} : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ be the symmetrization map. As $\text{symm} \otimes Id : S(\mathfrak{p}) \otimes U(\mathfrak{k}) \rightarrow U(\mathfrak{g})$ is a linear bijection and $\text{symm}(S(\mathfrak{p})) = \text{symm}(\mathcal{H}) \otimes \text{symm}(\mathcal{J})$, $\text{symm}(\mathcal{H}) \otimes V_\tau \cong I_{P,\sigma,\nu}$ as K modules.

Let (ξ, V_ξ) be an irreducible representation of K that occurs in $\mathcal{H} \otimes V_\tau$. Let $n(\xi)$ be the multiplicity of $V_\tau|_{{}^0M}$ in V_ξ . By Frobenius Reciprocity, the multiplicity of ξ type in $\mathcal{H} \otimes V_\tau$ is $n(\xi)$. Let $T_1^\xi, \dots, T_{n(\xi)}^\xi$ be a basis of $\text{Hom}_{{}^0M}(V_\tau, V_\xi)$ and let $\epsilon_1^\xi, \dots, \epsilon_{n(\xi)}^\xi$ be a basis of $\text{Hom}_K(V_\xi, \text{symm}(\mathcal{H}) \otimes V_\tau) \cong \text{Hom}_K(V_\xi, U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^K U(\mathfrak{k})} V_{\tau,\nu})$. Let $\mu_{\tau,\nu} : U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^K U(\mathfrak{k})} V_{\tau,\nu} \rightarrow I_{P,\sigma,\nu}$ be a (\mathfrak{g}, K) module homomorphism where the first factor acts on

the second as a differential operator $\pi_{P,\sigma,\nu}$ (cf. 11.3.6 [RRG II]), and define $R_\nu : \text{symm}(\mathcal{H}) \otimes V_\tau \longrightarrow V_\tau$ by $R_\nu(Z) := \mu_{\tau,\nu}(Z)(e)$ where $e \in G$ is the identity element. Every map defined in this paragraph intertwines 0M action, hence $R_\nu \circ \epsilon_i^\xi \circ T_j^\xi$ also for all i and j .

Definition 3.1.

- Define by $P^\xi(\nu)$ the $n(\xi)$ by $n(\xi)$ matrix where $(P^\xi(\nu))_{i,j}$ is the polynomial in ν in which $R_\nu \circ \epsilon_i^\xi \circ T_j^\xi$ acts on V_τ .
- Define by P^ξ the $n(\xi)$ by $n(\xi)$ matrix $P^\xi(\nu)$ as an element of $U(\mathfrak{a}) = S(\mathfrak{a}) \cong \mathcal{O}(\mathfrak{a}^*)$.
- Define by p_ξ and $p_\xi(\nu)$ the determinants of P^ξ and $P^\xi(\nu)$ respectively.

Remark 3.2 The definition of P^ξ depends on the choice of a small K type if $I_{P,\sigma,\nu}$ admits more than one small K type.

4. STRUCTURAL RESULTS

Let $\mathfrak{g}_\mathbb{R}$ be any of the split real form of simply laced, simple Lie type of rank ≥ 2 or let $\mathfrak{g}_\mathbb{R} = \mathfrak{gl}(n, \mathbb{R})$ for $n \geq 3$ in which case G is the metlinear group $\widetilde{GL}(n, \mathbb{R})$. In this section, we derive structural results for certain K modules. In 4.1, we analyze the structure of \mathcal{H}_α and $\mathcal{H}_\alpha \otimes V_\tau$ for the rank one subgroup G_α of G where α is a positive root of $\mathfrak{g}_\mathbb{R}$ and \mathcal{H}_α is \mathcal{H} for G_α . In 4.2, we give an exposition of the irreducible $Pin(n)$ modules that can be found in 5.5.5 of [GW]. In 4.3, we prove Frobenius Reciprocity statements in our context for the K module V_ξ that occurs in $\mathcal{H} \otimes V_\tau$.

4.1. Let $\mathfrak{g}_\mathbb{R} = \mathfrak{a}_\mathbb{R} \oplus \bigoplus_{\phi \in \Phi(\mathfrak{g}_\mathbb{R}, \mathfrak{a}_\mathbb{R})} \mathfrak{g}_\mathbb{R}^\phi$ be the root space decomposition. For α a positive root of $\mathfrak{g}_\mathbb{R}$, let $\mathfrak{g}_{\mathbb{R}\alpha} = \mathfrak{a}_\mathbb{R} \oplus \mathfrak{g}_\mathbb{R}^\alpha \oplus \mathfrak{g}_\mathbb{R}^{-\alpha}$. As $\mathfrak{g}_\mathbb{R}$ is split, $[\mathfrak{g}_{\mathbb{R}\alpha}, \mathfrak{g}_{\mathbb{R}\alpha}] \cong \mathfrak{sl}(2, \mathbb{R})$. Let $h_\alpha \in [\mathfrak{g}_{\mathbb{R}\alpha}, \mathfrak{g}_{\mathbb{R}\alpha}]$ be such that $\alpha(h_\alpha) = 2$ and let $e_\alpha \in \mathfrak{g}_\mathbb{R}^\alpha$ be such that $[e_\alpha, -\theta(e_\alpha)] = h_\alpha$. $(h_\alpha, e_\alpha, -\theta(e_\alpha))$ is an S triple.

Definition 4.1. Let α be a positive root of $\mathfrak{g}_\mathbb{R}$. $t_\alpha := i(e_\alpha + \theta(e_\alpha))$.

For a positive root α of $\mathfrak{g}_\mathbb{R}$, let $y_\alpha = e_\alpha - \theta(e_\alpha)$, $Z_\alpha = h_\alpha + iy_\alpha$, and $\overline{Z}_\alpha = h_\alpha - iy_\alpha$. Let Z_α^l and \overline{Z}_α^l be the l^{th} tensor power of Z_α and \overline{Z}_α respectively for $l \in \mathbb{Z}_{\geq 0}$. $t_\alpha \in \mathfrak{k}$, $[t_\alpha, Z_\alpha^l] = -2lZ_\alpha^l$, and $[t_\alpha, \overline{Z}_\alpha^l] = 2l\overline{Z}_\alpha^l$. Let $\exp : \mathfrak{g}_\mathbb{R} \rightarrow G$ be the exponential map. Let G_α be the rank one subgroup of G generated by 0M and the connected subgroup of G with Lie algebra $\mathfrak{g}_{\mathbb{R}\alpha}$. Let K_α be the subgroup of K generated by $\exp(i\mathbb{R}t_\alpha)$ and 0M , the maximal compact subgroup of G_α . Let \mathcal{H}_α be the space

of harmonics on $\mathfrak{p}_\alpha = \mathfrak{a} \oplus \mathbb{C}y_\alpha$ for the group G_α . \mathcal{H}_α as a space is $\bigoplus_{l \geq 0} \mathbb{C}Z_\alpha^l \bigoplus \bigoplus_{l > 0} \mathbb{C}\overline{Z}_\alpha^l$.

Lemma 4.1. *Let α be a positive root of $\mathfrak{g}_\mathbb{R}$. $\text{Ad}({}^0M)|_{\mathfrak{t}_\alpha} = \{\pm 1\}$.*

Proof. As $Z \subset Z(G)$, it suffices to show $\text{Ad}({}^0M_{G_\mathbb{R}})|_{\mathfrak{t}_\alpha} = \{\pm 1\}$. First assume $\mathfrak{g}_\mathbb{R}$ is any other than $\mathfrak{gl}(n, \mathbb{R})$. Since $G_\mathbb{C}$ is simply connected, the connected subgroup of $G_\mathbb{C}$ with Lie algebra $[\mathfrak{g}_\beta, \mathfrak{g}_\beta]$ is isomorphic to $SL(2, \mathbb{C})$ for $\beta \in \Phi^+$ where $\mathfrak{g}_\beta = \mathfrak{g}_{\mathbb{R}\beta} \otimes \mathbb{C}$. Since $G_\mathbb{R}$ is split, the connected subgroup G^β of $G_\mathbb{R}$ with Lie algebra $[\mathfrak{g}_{\mathbb{R}\beta}, \mathfrak{g}_{\mathbb{R}\beta}]$ is isomorphic to $SL(2, \mathbb{R})$ for $\beta \in \Phi^+$. Let $i_\beta : SL(2, \mathbb{R}) \rightarrow G^\beta$ be the isomorphism and let $\check{\beta}$ be the coroot to $\beta \in \Phi^+$. ${}^0M_{G_\mathbb{R}}$ is generated by the elements $\exp(\pi i \check{\beta}) = i_\beta(-\text{Id}) \in G^\beta$ for $\beta \in \Phi^+$. Since $\text{rank}(\mathfrak{g}_\mathbb{R}) \geq 2$ there exists a root β such that $\alpha(\check{\beta}) > 0$ hence $\alpha(\check{\beta}) = 1$. For $\widetilde{\mathfrak{g}_\mathbb{R}} = \widetilde{\mathfrak{gl}(n, \mathbb{R})}$, ${}^0M_{G_\mathbb{R}} \cong {}^0M_{SL(n, \mathbb{R})} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, restriction to $SL(n, \mathbb{R})$ proves the Lemma in this case. \square

Lemma 4.2. *Let α be a positive root of $\mathfrak{g}_\mathbb{R}$ and let τ be a small K type. The weights of t_α on V_τ are $\pm \frac{1}{2}$.*

Proof. Consider first the cases $G = \widetilde{SL(n, \mathbb{R})}$ and $G = \widetilde{GL(n, \mathbb{R})}$ where $K = Spin(n)$ and $K = Pin(n)$ respectively. For $K = Spin(n)$, (τ, V_τ) is the spin representation or either of the two half spin representation of $Spin(n)$ depending on the parity of n by Theorem 1. For $K = Pin(n)$, (τ, V_τ) is the pin representation or either of the two pin representation of $Pin(n)$ depending on the parity of n by Theorem 2.1. Hence the weights of t_α on V_τ are $\pm \frac{1}{2}$.

For G other than $\widetilde{SL(n, \mathbb{R})}$ and $\widetilde{GL(n, \mathbb{R})}$, recall from the proof of Theorem 1 the embedding $\tilde{i} : \widetilde{SL(n, \mathbb{R})} \hookrightarrow G$ or $\tilde{i} : \widetilde{GL(n, \mathbb{R})} \hookrightarrow G$ for appropriate n such that $\tilde{i}({}^0M_{\widetilde{SL(n, \mathbb{R})}}) \subseteq {}^0M_G$ or $\tilde{i}({}^0M_{\widetilde{GL(n, \mathbb{R})}}) \subseteq {}^0M_G$. The containment is an equality for all other than the case of D_n where ${}^0M_G \cong {}^0M_{\widetilde{SL(n, \mathbb{R})}} \times \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}/2\mathbb{Z}$ can be chosen to act trivially on V_τ (cf. proof of Theorem 1). In conclusion, V_τ restricted to either $\tilde{i}({}^0M_{\widetilde{SL(n, \mathbb{R})}})$ or $\tilde{i}({}^0M_{\widetilde{GL(n, \mathbb{R})}})$ is irreducible, hence V_τ is small as a $Spin(n)$ or a $Pin(n)$ module. There is a simple root β of $\mathfrak{g}_\mathbb{R}$ that is a simple root of $\text{Lie}(\tilde{i}(\widetilde{SL(n, \mathbb{R})}))$ or $\text{Lie}(\tilde{i}(\widetilde{GL(n, \mathbb{R})}))$. By the result of $\widetilde{SL(n, \mathbb{R})}$ and $\widetilde{GL(n, \mathbb{R})}$ above, the Lemma is proved for β . As $\mathfrak{g}_\mathbb{R}$ is simply laced, the positive roots are conjugates by elements of the Weyl group $W(A) = N_K(A)/Z_K(A)$. Hence the t_α s are also conjugates by certain representatives of the Weyl group elements (cf. Lemma 4.1) for positive

roots α . Therefore, the Lemma is proved for any positive root α of $\mathfrak{g}_{\mathbb{R}}$. \square

For a positive root α of $\mathfrak{g}_{\mathbb{R}}$, let ${}^0M_{\alpha}^{\pm}$ be the subsets of 0M whose elements act on t_{α} by ± 1 respectively. For a positive root α of $\mathfrak{g}_{\mathbb{R}}$, let $V_{\tau,\alpha}^{\pm}$ be the subspaces of V_{τ} that consist of t_{α} weight vectors of weight $\pm \frac{1}{2}$ respectively.

Lemma 4.3. *For a positive root α of $\mathfrak{g}_{\mathbb{R}}$, let $V_{\gamma_{\alpha}}$ be an irreducible nontrivial K_{α} module that occurs in \mathcal{H}_{α} . There exist exactly two t_{α} weights on $V_{\gamma_{\alpha}}$, and they are negatives of each other.*

Proof. Consider the t_{α} weight vector of weight $2l$, $\overline{Z_{\alpha}^l}$. 0M is a group of finite order that centralizes \mathfrak{a} and ${}^0M_{\alpha}^{\pm}$ act by ± 1 on t_{α} by Lemma 4.1. Therefore, ${}^0M_{\alpha}^{+}$ fixes $\overline{Z_{\alpha}^l}$ and ${}^0M_{\alpha}^{-}$ moves $\overline{Z_{\alpha}^l}$ to Z_{α}^l . Hence the weights are $2l$ and $-2l$. \square

Lemma 4.4. *For a positive root α of $\mathfrak{g}_{\mathbb{R}}$, let $V_{\xi_{\alpha}}$ be an irreducible K_{α} module that occurs in $\mathcal{H}_{\alpha} \otimes V_{\tau}$. There exist exactly two t_{α} weights on $V_{\xi_{\alpha}}$, and they are negatives of each other.*

Proof. Let $v \in V_{\xi_{\alpha}}$ be a t_{α} weight vector of weight c , which is nonzero as it is in the form of $2j \pm \frac{1}{2}$, because the only weights of t_{α} on V_{τ} are $\pm \frac{1}{2}$. If $m \in {}^0M$, $t_{\alpha}.m.v = m.m^{-1}.t_{\alpha}.m.m^{-1}.m.v = \pm m.t_{\alpha}.v = \pm c.m.v$. Since 0M acts irreducibly on $V_{\xi_{\alpha}}$ the result follows. \square

Theorem 4.5. *For a positive root α of $\mathfrak{g}_{\mathbb{R}}$, let $V_{\gamma_{\alpha}}$ be an irreducible K_{α} module that occurs in \mathcal{H}_{α} . $V_{\gamma_{\alpha}}$ as a space is the span of $\{\overline{Z_{\alpha}^l}, Z_{\alpha}^l\}$ for some l . Moreover, 0M invariant elements are $\mathbb{C}(\overline{Z_{\alpha}^l} + Z_{\alpha}^l)$.*

Proof. The statement follows from Lemma 4.3. \square

Theorem 4.6. *For a positive root α of $\mathfrak{g}_{\mathbb{R}}$, let $V_{\xi_{\alpha}}$ be an irreducible K_{α} module that occurs in $\mathcal{H}_{\alpha} \otimes V_{\tau}$. $V_{\xi_{\alpha}}$ as a space is either $(\overline{Z_{\alpha}^l} \otimes V_{\tau,\alpha}^{+}) \oplus (Z_{\alpha}^l \otimes V_{\tau,\alpha}^{-})$ or $(\overline{Z_{\alpha}^l} \otimes V_{\tau,\alpha}^{-}) \oplus (Z_{\alpha}^l \otimes V_{\tau,\alpha}^{+})$ for some l .*

Proof. Consider $\overline{Z_{\alpha}^l} \otimes v$ for some l and $v \in V_{\tau,\alpha}^{+}$. By Lemma 4.4, K_{α} module generated by this element is contained in $(\overline{Z_{\alpha}^l} \otimes V_{\tau,\alpha}^{+}) \oplus (Z_{\alpha}^l \otimes V_{\tau,\alpha}^{-})$. Since the dimensions of the two spaces are equal the inclusion is an equality. We argue similarly if we start with a vector in $V_{\tau,\alpha}^{-}$. \square

Corollary 4.7. *Let α be a positive root of $\mathfrak{g}_{\mathbb{R}}$. There is a unique dominant t_{α} weight on an irreducible K_{α} module $V_{\gamma_{\alpha}}$ that occurs in \mathcal{H}_{α} and a unique dominant t_{α} weight on an irreducible K_{α} module $V_{\xi_{\alpha}}$ that occurs in $\mathcal{H}_{\alpha} \otimes V_{\tau}$.*

Proof. The two statements are direct consequences of Theorem 4.5 and Theorem 4.6 respectively. \square

4.2. We give an exposition of the irreducible $Pin(n)$ modules that can be found in 5.5.5 of [GW]. The exposition is given for the groups $O(n)$ and $SO(n)$ in [GW]; however, the exposition is also true for the groups $Pin(n)$ and $Spin(n)$. If $n = 2k + 1$ is odd, let $g_0 = -Id \in O(2k + 1)$. If $n = 2k$ is even, let $g_0 \in O(2k)$ be the diagonal matrix whose entries are all 1 except for last $g_{0,2k,2k} = -1$. Let $p : Pin(n) \rightarrow O(n)$ be the covering homomorphism and let ζ be a choice of $p^{-1}(g_0)$. Let (π_λ, V_λ) be the irreducible representation of $Spin(n)$ with highest weight λ and let $(\rho_\lambda, V_\lambda)$ be the induced representation $Ind_{Spin(2k)}^{Pin(2k)}(\pi_\lambda)$. Let $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition of $Lie(Spin(n))_{\mathbb{C}}$.

Theorem 4.8. (Theorem 5.5.23 in [GW]) *The irreducible regular representations of $Pin(2k + 1)$ are of the form $(\pi_\lambda^\epsilon, V_\lambda^\epsilon)$, where $(\pi_\lambda^\epsilon, V_\lambda^\epsilon)$ restricted to $Spin(2k + 1)$ is the highest weight representation (π_λ, V_λ) , and ζ acts on V_λ^ϵ by ϵI where $\epsilon = \pm$.*

Theorem 4.9. (Theorem 5.5.24 in [GW]) *Let $k \geq 2$. The irreducible representation (σ, W) of $Pin(2k)$ is one of the following two types.*

- Suppose $\dim W^{\mathfrak{n}^+} = 1$ and \mathfrak{h} acts by the weight λ on $W^{\mathfrak{n}^+}$. $(\sigma, W) \cong (\pi_\lambda^\epsilon, V_\lambda^\epsilon)$ where $(\pi_\lambda^\epsilon, V_\lambda^\epsilon)$ restricted to $Spin(2k)$ is the highest weight representation (π_λ, V_λ) , and ζ acts on $W^{\mathfrak{n}^+}$ by ϵI where $\epsilon = \pm$.
- Suppose $\dim W^{\mathfrak{n}^+} = 2$. Then \mathfrak{h} has two distinct weights λ and $\zeta \cdot \lambda$ on $W^{\mathfrak{n}^+}$, and $(\sigma, W) \cong (\rho_\lambda, V_\lambda)$.

4.3. For an irreducible representation (δ, V_δ) of a subgroup of G , let $d(\delta) = \dim V_\delta$. If (δ, V_δ) is an irreducible K module that occurs in \mathcal{H} , let $l(\delta) = \dim V_\delta^{0M}$ and if (δ, V_δ) is an irreducible K module that occurs in $\mathcal{H} \otimes V_\tau$, let $n(\delta) = \dim Hom_K(V_\delta, \mathcal{H} \otimes V_\tau)$.

Lemma 4.10. *Let ξ be a K type that occurs in $\mathcal{H} \otimes V_\tau$, and let $\gamma_1, \dots, \gamma_N$ be the distinct K types that occur in \mathcal{H} such that $V_\xi \subset V_{\gamma_j} \otimes V_\tau$ for all $j = 1, \dots, N$. $n(\xi) = \sum_{j=1}^N l(\gamma_j)$.*

Proof. For G other than $\widetilde{GL(n, \mathbb{R})}$, V_ξ has multiplicity one in $V_{\gamma_j} \otimes V_\tau$ as V_τ is multiplicity free (Proposition 3.2 [Ku]). For $G = \widetilde{GL(n, \mathbb{R})}$, we restrict to $SL(n, \mathbb{R})$ and use the branching of irreducible $Pin(n)$ modules to $Spin(n)$ (cf. 4.2) to deduce that V_ξ has multiplicity one in $V_{\gamma_j} \otimes V_\tau$. Each of V_{γ_j} has multiplicity $l(\gamma_j)$ in \mathcal{H} by Frobenius Reciprocity. Hence $n(\xi) = \sum_{j=1}^N l(\gamma_j)$. \square

Lemma 4.11. *If V_ξ is an irreducible K module that occurs in $\mathcal{H} \otimes V_\tau$, $V_\xi|_{^0M} = \bigoplus_{j=1}^{\frac{d(\xi)}{d(\tau)}} V_{\tau_j}$ where $V_{\tau_j} \cong V_\tau$ as 0M modules.*

Proof. For $G = \widetilde{SL(n, \mathbb{R})}$, let η be the nontrivial element of $Z = \mathbb{Z}/2\mathbb{Z}$. $\mathbb{C}[^0M_{\widetilde{SL(n, \mathbb{R})}}]/\langle \eta + 1 \rangle$ is isomorphic to the simple matrix algebra $M_{2^{\frac{n}{2}}}(\mathbb{C})$ for n odd and a direct sum of two simple matrix algebras $M_{2^{\frac{n-1}{2}}}(\mathbb{C}) \oplus M_{2^{\frac{n-1}{2}}}(\mathbb{C})$ for n even (cf. proof of Theorem 1). Hence we have the statement of the Lemma if n is odd. If n is even, V_{τ_j} is either of the two half spin representations restricted to $^0M_{\widetilde{SL(n, \mathbb{R})}}$. Let ω be a choice of $p^{-1}(-Id)$ where $p : Spin(n) \rightarrow SO(n)$ is the covering homomorphism. ω distinguishes the two representations as $^0M_{\widetilde{SL(n-1, \mathbb{R})}}$ does not. ω acts trivially on \mathcal{H} as it is central in K and is an element of $^0M_{\widetilde{SL(n, \mathbb{R})}}$. Therefore ω acts on the entire space $\mathcal{H} \otimes V_\tau$ as it does on V_τ , hence we have the statement of the Lemma for $\widetilde{SL(n, \mathbb{R})}$. For $G = GL(n, \mathbb{R})$ the metaleinear group, the statement of the Lemma is proved similarly as $^0M_{\widetilde{GL(n, \mathbb{R})}} \cong ^0M_{\widetilde{SL(n+1, \mathbb{R})}}$ (cf. proof of Theorem 1).

For $G = \widetilde{Spin(n, n)_0}$, $^0M_G \cong ^0M_{\widetilde{SL(n, \mathbb{R})}} \times \mathbb{Z}/2\mathbb{Z}$ where the $\mathbb{Z}/2\mathbb{Z} \subset K$ and can be chosen to act trivially on \mathcal{H} and V_τ (cf. proof of Theorem 1). Therefore, $(\mathcal{H} \otimes V_\tau)|_{^0M_G}$ decomposes in the same way as $(\mathcal{H} \otimes V_\tau)|_{^0M_{\widetilde{SL(n, \mathbb{R})}}}$ where the factor $^0M_{\widetilde{SL(n, \mathbb{R})}}$ is embedded diagonally in $K = Spin(n) \times Spin(n)$. If n is odd, the statement of the Lemma is proved similarly as in the case $\widetilde{SL(n, \mathbb{R})}$. If n is even, there are two possibilities for $V_\tau|_{^0M_G}$ as in the case of $\widetilde{SL(n, \mathbb{R})}$. Let ω be a choice of $p^{-1}(-Id \times -Id)$ where p is the covering homomorphism $p : Spin(n) \times Spin(n) \rightarrow SO(n) \times SO(n)$. ω distinguishes the two representations as $^0M_{\widetilde{SL(n-1, \mathbb{R})}}$ does not. ω acts trivially on \mathcal{H} as it is central in K and is an element of $^0M_{\widetilde{Spin(n, n)_0}}$. Therefore ω acts on the entire space $\mathcal{H} \otimes V_\tau$ as it does on V_τ , hence we have the statement of the Lemma for $\widetilde{Spin(n, n)_0}$.

For G of type E_7 , 0M is isomorphic to $^0M_{\widetilde{SL(8, \mathbb{R})}}$. $Z(G) \cong \mathbb{Z}/4\mathbb{Z}$. $Z(G) \subset Z(K)$ by Theorem 7.2.5 of [HAHS]. As $Z(K) = \{zId \mid z^8 = 1\}$, $Z(G)$ in K consist of $\pm Id$ and $\pm i * Id$. As $Z(G) \subset ^0M$, $i * Id \in ^0M$ acts trivially on \mathcal{H} . $i * Id$ distinguishes the two 0M irreducible representations \mathbb{C}^8 and $(\mathbb{C}^8)^*$ as it acts by different signs on the two. $i * Id$ acts on $\mathcal{H} \otimes V_\tau$ as it does on V_τ . Therefore, the Lemma is proved similarly as above.

For G of type E_6 and E_8 , 0M is isomorphic to ${}^0M_{\widetilde{SL(n, \mathbb{R})}}$ for n odd (cf. proof of Theorem 1). Therefore the Lemma is proved similarly as the above case for type $\widetilde{SL(n, \mathbb{R})}$. \square

5. PRODUCT FORMULA FOR p_ξ

Let $\mathfrak{g}_{\mathbb{R}}$ be any of the split real form of simply laced, simple Lie type of rank ≥ 2 . Let G be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Recall the definition of P^ξ for a K type ξ that occurs in $I_{P, \sigma, \nu}$. The definition of P^ξ depends on the choice of a small K type if $I_{P, \sigma, \nu}$ admits more than one (cf. section 3); however, for G , $I_{P, \sigma, \nu}$ admits only one small K type (cf. Theorem 1 and proof of Lemma 4.11). In this section, we give a product formula for p_ξ over the rank one subgroups of G corresponding to the positive roots. The proof of the product formula consists of two parts, degree argument and divisibility argument. In 5.1, we give a comparison of t_α weights on relevant vectors for α a positive root that is necessary for the degree argument. In 5.2, we give the divisibility argument. In 5.3, we give the proof of the product formula for p_ξ .

Let V_γ be an irreducible K module that occurs in \mathcal{H} , and let V_ξ be an irreducible K module that occurs in $\mathcal{H} \otimes V_\tau$. For a positive root α of $\mathfrak{g}_{\mathbb{R}}$, let $\text{Span } K_\alpha \cdot V_\gamma^{0M} = \bigoplus_{j=1}^{l(\gamma)} W_j^\alpha$ be a decomposition into irreducible K_α submodules where $l(\gamma) = \dim V_\gamma^{0M}$ and $\dim W_j^\alpha \cap V_\gamma^{0M} = 1$ for $j = 1, \dots, l(\gamma)$ (cf. Theorem 2.3.7 [Kos]). Let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}^\alpha$ be a decomposition into K_α submodules such that $V_{\tau_j}^\alpha$ is an irreducible K_α module with $V_{\tau_j}|_{0M} \cong V_\tau$ for $j = 1, \dots, n(\xi)$ by Lemma 4.11.

Definition 5.1.

- Let $\delta_{\alpha, j}^\gamma$ be the unique dominant t_α weight on W_j^α for $j = 1, \dots, l(\gamma)$ by Corollary 4.7.
- Let $\delta_{\alpha, j}^\xi$ be the unique dominant t_α weight on $V_{\tau_j}^\alpha$ for $j = 1, \dots, n(\xi)$ by Corollary 4.7.

5.1. Let V_Γ be an irreducible $\text{Pin}(n)$ module that occurs in \mathcal{H} the space of harmonics on \mathfrak{p} for the metlinear group $\widetilde{GL(n, \mathbb{R})}$. \mathcal{H} is also that of $\widetilde{SL(n, \mathbb{R})}$. Let V_γ be an irreducible $\text{Spin}(n)$ module that occurs in $V_\Gamma|_{\text{Spin}(n)}$. Recall the definition of ζ from 4.2.

Theorem 5.1. *Let $n = 2k + 1$. ζ acts trivially on V_Γ , and there is no difference between V_γ and V_Γ .*

Proof. As the modules in the Theorem are submodules of \mathcal{H} , we may assume the modules are $SO(n)$ and $O(n)$ modules. Recall the definition of g_0 from 4.2. $g_0 \in Z(O(n))$ and $g_0 \in {}^0M_{GL(n, \mathbb{R})}$. As $V_\Gamma \subset \mathcal{H}$, g_0 must act trivially on V_Γ , hence there is no difference between V_γ and V_Γ by Theorem 4.8. \square

Theorem 5.2. *Let $n = 2k$, and let (m_1, \dots, m_k) be the highest weight of V_γ .*

- *If $m_k \neq 0$, $\dim V_\gamma^{0M_{SL(n, \mathbb{R})}} = \dim V_\Gamma^{0M_{GL(n, \mathbb{R})}}$, $l(\gamma) = l(\Gamma)$ and $\delta_{\alpha, j}^\gamma = \delta_{\alpha, j}^\Gamma$ for all j after reordering.*
- *If $m_k = 0$, let $V_\Gamma = V_\gamma^\epsilon$ where $\epsilon = \pm$ is the signature of ζ on the highest weight vector by Theorem 4.9. $V_\gamma^{0M_{SL(n, \mathbb{R})}} = V_\gamma^{+0M_{GL(n, \mathbb{R})}} \oplus V_\gamma^{-0M_{GL(n, \mathbb{R})}}$, hence $\delta_{\alpha, 1}^\gamma, \dots, \delta_{\alpha, l(\gamma)}^\gamma$ is a disjoint union of those of $V_\gamma^{+0M_{GL(n, \mathbb{R})}}$ and $V_\gamma^{-0M_{GL(n, \mathbb{R})}}$.*

Proof. As the modules in the Theorem are submodules of \mathcal{H} , we may assume the modules are $SO(n)$ and $O(n)$ modules. Recall the definition of g_0 from 4.2.

Let us assume first $m_k \neq 0$. g_0 swaps the two $SO(2k)$ highest weight modules of highest weights (m_1, \dots, m_k) and $(m_1, \dots, -m_k)$. As g_0 commutes with ${}^0M_{SL(n, \mathbb{R})}$, g_0 gives a bijection of ${}^0M_{SL(n, \mathbb{R})}$ -invariants in the $SO(2k)$ highest weight module of highest weight (m_1, \dots, m_k) with ${}^0M_{SL(n, \mathbb{R})}$ -invariants in the $SO(2k)$ highest weight module of highest weight $(m_1, \dots, -m_k)$. Hence, $\dim V_\gamma^{0M_{SL(n, \mathbb{R})}} = \dim V_\Gamma^{0M_{GL(n, \mathbb{R})}}$ as ${}^0M_{GL(n, \mathbb{R})}$ is generated by ${}^0M_{SL(n, \mathbb{R})}$ and g_0 . As g_0 leaves invariant t_α^2 , the statement of the t_α weights is also proved by Lemma 4.3.

Assume $m_k = 0$. Let $v(m_1, \dots, m_k)$ be the highest weight vector, and let $v_1, \dots, v_{l(\gamma)}$ be a basis of $V_\gamma^{0M_{SL(n, \mathbb{R})}}$ such that g_0 acts on v_j by ± 1 for all j , which is possible as $g_0^2 = Id$ and g_0 commutes with ${}^0M_{SL(n, \mathbb{R})}$. Denote by $v_1^+, \dots, v_{l(\gamma)}^+$ and $v_1^-, \dots, v_{l(\gamma)}^-$ the above basis thought of being in V_γ^+ and V_γ^- respectively. The only difference between V_γ^+ and V_γ^- is the action of g_0 . Denote by ϵ the action of g_0 on $v_1^+, \dots, v_{l(\gamma)}^+$. g_0 acts by $-\epsilon$ on $v_1^-, \dots, v_{l(\gamma)}^-$. Indeed, $v_j = X_j.v(m_1, \dots, m_k)$ where $X_j \in U(\mathfrak{n}^-)$. As g_0 acts by different signatures on $v(m_1, \dots, m_k)$ for V_γ^+ and V_γ^- , the statement is now proved. \square

Denote by (T, V_T) the small $Pin(n)$ type and denote by V_Ξ an irreducible $Pin(n)$ module that occurs in $\mathcal{H} \otimes V_T$ for convenience. Let V_ξ be an irreducible $Spin(n)$ module that occurs in $V_\Xi|_{Spin(n)}$. $V_\xi \subseteq \mathcal{H} \otimes V_\tau$ where V_τ is a small $Spin(n)$ representation. Let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}^\alpha$ be a

decomposition into irreducible K_α submodules such that $V_{\tau_j^\alpha} \cong V_\tau$ as ${}^0M_{\widetilde{SL(n, \mathbb{R})}}$ modules for all j by Lemma 4.11.

Theorem 5.3. *If $n = 2k$, $V_\Xi|_{{}^0M_{\widetilde{GL(n, \mathbb{R})}}} = \bigoplus_{j=1}^{n(\xi)} (V_{\tau_j^\alpha} \oplus \zeta \cdot V_{\tau_j^\alpha})$. If $n = 2k + 1$, $V_\Xi|_{{}^0M_{\widetilde{GL(n, \mathbb{R})}}} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j^\alpha}$. In either case, $\delta_{\alpha, j}^\xi = \delta_{\alpha, j}^\Xi$ for all j after reordering.*

Proof. As ${}^0M_{\widetilde{GL(n, \mathbb{R})}}$ is generated by ${}^0M_{\widetilde{SL(n, \mathbb{R})}}$ and ζ , the Theorem is proved by Theorem 4.8, Theorem 4.9, and Lemma 4.11. \square

Let ξ be a K type that occurs in $\mathcal{H} \otimes V_\tau$ and let $\gamma_1, \dots, \gamma_N$ be the distinct K types that occur in \mathcal{H} such that $V_\xi \subset V_{\gamma_j} \otimes V_\tau$ for all $j = 1, \dots, N$. For α a positive root of $\mathfrak{g}_{\mathbb{R}}$, let $\delta_{\alpha, 1}^\xi, \dots, \delta_{\alpha, n(\xi)}^\xi$ be the t_α weights on V_ξ and let $\delta_{\alpha, 1}, \dots, \delta_{\alpha, \sum_{j=1}^N l(\gamma_j)}$ be the t_α weights on $V_{\gamma_1}, \dots, V_{\gamma_N}$ defined in 5.1. $n(\xi) = \sum_{j=1}^N l(\gamma_j)$ by Lemma 4.10.

Theorem 5.4. *$\delta_{\alpha, 1}^\xi, \dots, \delta_{\alpha, n(\xi)}^\xi$ can be reordered so that $\delta_{\alpha, j} = \delta_{\alpha, j}^\xi \pm \frac{1}{2}$ for all $j = 1, \dots, n(\xi)$.*

We will need the following Lemma for the proof of Theorem 5.4.

Lemma 5.5. *The comparison of t_α weights in Theorem 5.4 for $\widetilde{SL(n, \mathbb{R})}$ implies that for $\widetilde{GL(n, \mathbb{R})}$.*

Proof. The space of Harmonics \mathcal{H} on \mathfrak{p} are the same for both $\widetilde{SL(n, \mathbb{R})}$ and $\widetilde{GL(n, \mathbb{R})}$.

Assume first n is odd. A small $Pin(n)$ representation V_T is the small $Spin(n)$ representation V_τ if we restrict from $Pin(n)$ to $Spin(n)$. By Theorem 5.1, the restriction of $Pin(n)$ to $Spin(n)$ does not change the assumptions of the modules in Theorem 5.4. Given an irreducible $Pin(n)$ module that occurs in \mathcal{H} , the t_α weights of interest do not change restricted to $Spin(n)$ by Theorem 5.1. Given an irreducible $Pin(n)$ module that occurs in $\mathcal{H} \otimes V_T$, the t_α weights of interest do not change restricted to $Spin(n)$ by Theorem 5.3. Therefore, the comparison of the t_α weights for the group $Pin(n)$ reduce to the comparison of t_α weights for the group $Spin(n)$ by restriction of $Pin(n)$ to $Spin(n)$.

Assume now n is even. The small $Pin(n)$ representation V_T is a direct sum of the two half spin representations V_τ and \overline{V}_τ if we restrict from $Pin(n)$ to $Spin(n)$. Let $\Gamma_1, \dots, \Gamma_M$ be the distinct $Pin(n)$ types that occur in \mathcal{H} such that $V_\Xi \subseteq V_{\Gamma_j} \otimes V_T$. Let us restrict V_{Γ_j} to $Spin(n)$. By Theorem 4.9, if $\dim V_{\Gamma_j}^{n^+} = 2$, V_{Γ_j} is a direct sum of two irreducible

$Spin(n)$ modules with last entries of the highest weights nonzero and negatives of each other, and if $\dim V_{\Gamma_j}^{n^+} = 1$, V_{Γ_j} is irreducible as a $Spin(n)$ module. For each j , let V_{γ_j} be the choice of the irreducible $Spin(n)$ module that occurs in $V_{\Gamma_j}|_{Spin(n)}$ with last entry of the highest weight nonnegative, and reorder so that $\gamma_1, \dots, \gamma_N$ are distinct $Spin(n)$ types. $N \leq M$ as there may be j such that V_{γ_j} occurs twice with different g_0 signature on $V_{\gamma_j}^{n^+}$. Let V_{ξ} be the choice of the irreducible $Spin(n)$ module that occurs in $V_{\Xi}|_{Spin(n)}$ with last entry of the highest weight positive. Without loss of generality, assume $V_{\xi} \subseteq \mathcal{H} \otimes V_{\tau}$. $\gamma_1, \dots, \gamma_N$ are distinct $Spin(n)$ types that occur in \mathcal{H} such that $V_{\xi} \subseteq V_{\gamma_j} \otimes V_{\tau}$. Therefore, we can assume the statement of the t_{α} weights on these $Spin(n)$ modules. But by Theorem 5.2 and Theorem 5.3, the comparison of t_{α} weights for the modules of the group $Pin(n)$ is that of $V_{\xi}, V_{\gamma_1}, \dots, V_{\gamma_N}$ of $Spin(n)$. Therefore, we have the result for n even. \square

Proof. (of Theorem 5.4)

The t_{α} weights of interest is independent of the choice of $\alpha \in \Phi^+$. Indeed, let ψ and ϕ be two positive roots. They are conjugates by an element of the Weyl group $N_K(A)/Z_K(A)$ hence t_{ψ} and t_{ϕ} are also, by certain representative of the above Weyl group element (cf. Lemma 4.1). Let s be such a representative. $sK_{\psi}s^{-1} = K_{\phi}$. Let $V_{\xi} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j^{\psi}}$ be a decomposition into K_{ψ} submodules such that each $V_{\tau_j^{\psi}}$ is an irreducible K_{ψ} module isomorphic to $V_{\tau}|_{^0M}$ when restricted to 0M . If $V_{\tau_j^{\phi}} = sV_{\tau_j^{\psi}}$, then $V_{\xi} = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j^{\phi}}$ is a decomposition into K_{ϕ} submodules such that each $V_{\tau_j^{\phi}}$ is an irreducible K_{ϕ} module isomorphic to $V_{\tau}|_{^0M}$ when restricted to 0M by Lemma 4.11.

We first prove the Theorem for $\widetilde{SL(n, \mathbb{R})}$.

Let $n = 3$. Let $\xi = \frac{p}{2}$ be the highest weight of V_{ξ} with p odd. If $p = 1$, there exists only one $V_{\gamma} \subseteq \mathcal{H}$ with $V_{\xi} \subseteq V_{\gamma} \otimes V_{\tau}$ the trivial representation, and the claim is true. If $p = 3$, there exists only one $V_{\gamma} \subseteq \mathcal{H}$ with $V_{\xi} \subseteq V_{\gamma} \otimes V_{\tau}$ the representation with highest weight 2. In this case, the weights are $\frac{1}{2}$ and $\frac{3}{2}$ for V_{ξ} and 0 and 2 for V_{γ} , hence the claim is also true. Suppose $p > 3$. Then, there exist exactly two such representations, $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$ with highest weights $\frac{p-1}{2}$ and $\frac{p+1}{2}$. The weights of interest on V_{ξ} are $\frac{1}{2}, \frac{3}{2}, \dots, \frac{p}{2}$. The weights of interest on $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$ are $0, 2, 4, \dots, \frac{p-1}{2}$ and $2, 4, \dots, \frac{p-1}{2}$ respectively if $\frac{p-1}{2}$ is even, and $2, 4, \dots, \frac{p-3}{2}$ and $0, 2, \dots, \frac{p+1}{2}$ respectively if $\frac{p+1}{2}$ is even. In the first case where $\frac{p-1}{2}$ is even, consider $(\frac{1}{2} - \frac{1}{2}), (\frac{3}{2} + \frac{1}{2}), (\frac{5}{2} - \frac{1}{2}), (\frac{7}{2} + \frac{1}{2}), \dots, (\frac{p-2}{2} + \frac{1}{2}), (\frac{p}{2} - \frac{1}{2})$, which are $0, 2, 2, 4, 4, \dots, \frac{p-1}{2}, \frac{p-1}{2}$. This is exactly the union

of the weights on $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$. In the second case where $\frac{p+1}{2}$ is even, consider $(\frac{1}{2} - \frac{1}{2}), (\frac{3}{2} + \frac{1}{2}), (\frac{5}{2} - \frac{1}{2}), (\frac{7}{2} + \frac{1}{2}), \dots, (\frac{p-2}{2} - \frac{1}{2}), (\frac{p}{2} + \frac{1}{2})$, which are $0, 2, 2, 4, 4, \dots, \frac{p-3}{2}, \frac{p-3}{2}, \frac{p+1}{2}$. This is again exactly the union of the weights on $V_{\gamma_{p_1}}$ and $V_{\gamma_{p_2}}$. Therefore the statement is proved for the case $n = 3$.

We now proceed by induction. Assume the Theorem for $\widetilde{SL(n, \mathbb{R})}$, and hence for $\widetilde{GL(n, \mathbb{R})}$ by Lemma 5.5. We prove the Theorem for $\widetilde{SL(n+1, \mathbb{R})}$. Recall from the proof of Theorem 1 the embedding $\tilde{i} : \widetilde{GL(n, \mathbb{R})} \hookrightarrow \widetilde{SL(n+1, \mathbb{R})}$ with $\tilde{i}({}^0M_{\widetilde{GL(n, \mathbb{R})}}) = {}^0M_{\widetilde{SL(n+1, \mathbb{R})}}$. We drop the notation \tilde{i} for simplicity. We can restate the condition $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ with $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$ where V_τ^* is the contragredient representation. The statement of the Theorem is true with the restated condition for $\widetilde{GL(n, \mathbb{R})}$ by the induction hypothesis.

Let $\gamma_1, \dots, \gamma_N$ be the distinct $Spin(n+1)$ types that occur in \mathcal{H} such that $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$, and let $\bigoplus_{j=1}^N \text{Span } Pin(n).V_{\gamma_j}^{0M} = \bigoplus_k W_k$ where each W_k is an irreducible $Pin(n)$ module. $V_\xi|_{Pin(n)} = \bigoplus_j V_{\xi_j}$ where each V_{ξ_j} occurs in $\mathcal{H} \otimes V_\tau$ by Lemma 4.11, with \mathcal{H} that of $\widetilde{GL(n, \mathbb{R})}$. $\bigoplus_k W_k \subseteq \bigoplus_j V_{\xi_j} \otimes V_\tau^*$ where each W_k occurs in \mathcal{H} of $\widetilde{GL(n, \mathbb{R})}$ as ${}^0M_{\widetilde{GL(n, \mathbb{R})}} = {}^0M_{\widetilde{SL(n+1, \mathbb{R})}}$. $V_{\xi_j} \otimes V_\tau^*$ decomposes into distinct $Pin(n)$ types as V_τ^* is multiplicity free (cf. proof of Lemma 4.10). Therefore, if $W_k \cong W_l$ with $k \neq l$, W_k and W_l cannot be contained in a single $V_{\xi_j} \otimes V_\tau^*$. This is important as the statement of the Theorem for $\widetilde{GL(n, \mathbb{R})}$ also assumes distinct Γ types. As the statement of the Theorem is true for $\widetilde{GL(n, \mathbb{R})}$ with the restated condition and the t_α weights of interest are the same after branching down to $Pin(n)$, the Theorem is proved for all positive roots α of $Lie(\widetilde{SL(n+1, \mathbb{R})})$ that restrict to a positive root of $Lie(\widetilde{GL(n, \mathbb{R})}) \subseteq Lie(\widetilde{SL(n+1, \mathbb{R})})$, hence for all positive roots of $Lie(\widetilde{SL(n+1, \mathbb{R})})$.

Let G be other than $\widetilde{SL(n, \mathbb{R})}$. Recall from the proof of Theorem 1 the embedding $\tilde{i} : S \hookrightarrow G$ with $\tilde{i}({}^0M_S) \subseteq {}^0M_G$ where $S \cong \widetilde{SL(n, \mathbb{R})}$ or $\widetilde{GL(n, \mathbb{R})}$ for appropriate n . Denote by K_S the maximal compact subgroup of S . We drop the notation \tilde{i} for simplicity. ${}^0M_S = {}^0M_G$ for all but the case of D_n where ${}^0M_G \cong {}^0M_{\widetilde{SL(n, \mathbb{R})}} \times \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}/2\mathbb{Z} \subset Z(K)$ and can be chosen to act trivially on the small K type (cf. proof of Theorem 1). Therefore, the small K representation V_τ is also a small K_S representation.

We can restate the condition $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$ as $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$. The statement of the Theorem is true with the restated condition for S . Let $\gamma_1, \dots, \gamma_N$ be the distinct K types that occur in \mathcal{H} such that $V_{\gamma_j} \subseteq V_\xi \otimes V_\tau^*$, and let $\bigoplus_{j=1}^N \text{Span } K_S.V_{\gamma_j}^{0M_G} = \bigoplus_k W_k$ where each W_k is an irreducible K_S module. $V_\xi|_{K_S} = \bigoplus_j V_{\xi_j}$ where each V_{ξ_j} occurs in $\mathcal{H} \otimes V_\tau$ by Lemma 4.11, with \mathcal{H} that of S . $\bigoplus_k W_k \subseteq \bigoplus_j V_{\xi_j} \otimes V_\tau^*$ where each W_k occurs in \mathcal{H} of S as ${}^0M_G = {}^0M_S$ or ${}^0M_G = {}^0M_S \times \mathbb{Z}/2\mathbb{Z}$ where the $\mathbb{Z}/2\mathbb{Z} \subset Z(K)$. $V_{\xi_j} \otimes V_\tau^*$ decomposes into distinct K_S types as V_τ^* is multiplicity free (cf. proof of Lemma 4.10). Therefore, if $W_k \cong W_l$ with $k \neq l$, W_k and W_l cannot be contained in a single $V_{\xi_j} \otimes V_\tau^*$. This is important as the statement of the Theorem for S also assumes distinct γ types. As the Theorem is true for S with the restated condition and as the t_α weights of interest are the same after branching down to K_S , the Theorem is proved for all positive roots α of $\text{Lie}(G)$ that restrict to a positive root of $\text{Lie}(S) \subseteq \text{Lie}(G)$, hence for all positive roots of $\text{Lie}(G)$. \square

5.2. Recall the notation of 4.1. For α a positive root of $\mathfrak{g}_\mathbb{R}$, let $\mathfrak{g}_\alpha = \mathbb{C}\theta(e_\alpha) \oplus \mathfrak{a} \oplus \mathbb{C}e_\alpha = \theta(\mathfrak{n}_\alpha) \oplus \mathfrak{a} \oplus \mathfrak{n}_\alpha$ be the triangular decomposition and let $\mathfrak{g}_\alpha = \mathbb{C}(e_\alpha + \theta(e_\alpha)) \oplus (\mathbb{C}(e_\alpha - \theta(e_\alpha)) \oplus \mathfrak{a}) = \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha$ be the Cartan decomposition. Let \mathcal{H}_α be the space of harmonics on \mathfrak{p}_α and let $\mathcal{J}_\alpha = S(\mathfrak{p}_\alpha)^{\mathfrak{k}_\alpha}$. For α simple, let $\mathfrak{n}^\alpha = \bigoplus_{\psi \in \Phi^+ - \{\alpha\}} \mathfrak{g}^\psi$. Let $\mathfrak{k}^\alpha = \bigoplus_{\psi \in \Phi^+ - \{\alpha\}} \mathbb{C}(e_\psi + \theta e_\psi)$ so that $\mathfrak{k} = \mathfrak{k}_\alpha \oplus \mathfrak{k}^\alpha$. $\mathfrak{g} = \mathfrak{n}^\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{k}^\alpha$ and \mathfrak{n}^α is a Lie subalgebra of \mathfrak{g} as α is simple.

Lemma 5.6. *For α a simple root of $\mathfrak{g}_\mathbb{R}$,*

$$U(\mathfrak{g}) = \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) U(\mathfrak{k}) \oplus \mathfrak{n}^\alpha U(\mathfrak{n}^\alpha) \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) U(\mathfrak{k})$$

Proof. By Proposition 2.4.1 of [Kos],

$$U(\mathfrak{g}) = U(\mathfrak{n}^\alpha) \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) \oplus U(\mathfrak{g}) \mathfrak{k}$$

Hence,

$$\begin{aligned} U(\mathfrak{g}) &= U(\mathfrak{n}^\alpha) \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) U(\mathfrak{k}) \\ &= \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) U(\mathfrak{k}) \oplus \mathfrak{n}^\alpha U(\mathfrak{n}^\alpha) \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) U(\mathfrak{k}) \end{aligned}$$

\square

As $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^\mathfrak{k}} V_\tau \cong I_{P,\sigma,\nu}$ as K modules (cf. 11.3.6 [RRG II]), we have the following K module isomorphisms by Lemma 5.6.

$$\begin{aligned} I_{P,\sigma,\nu} &\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^\mathfrak{k}} V_\tau \\ &\cong \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^\mathfrak{k}} V_\tau \\ &\quad \oplus \mathfrak{n}^\alpha U(\mathfrak{n}^\alpha) \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^\mathfrak{k}} V_\tau \end{aligned}$$

Let $\mathbb{C}[{}^0M]$ be the group algebra generated by 0M . Denote by $U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M]$ the smash product of $U(\mathfrak{k}_\alpha)$ with $\mathbb{C}[{}^0M]$, i.e. $U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M]$ has a $(U(\mathfrak{k}_\alpha), \mathbb{C}[{}^0M])$ action on $\text{symm}(\mathcal{H}_\alpha) \otimes V_\tau$ that is an analog of a (\mathfrak{g}, K) action. Let $I_\tau = U(\mathfrak{k}) \cap \ker \tau$. As 0M acts irreducibly on V_τ , $U(\mathfrak{k})/I_\tau \cong \text{End}(V_\tau) \cong (U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M]) / (\ker \tau \cap (U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M]))$.

For α simple, let

$$L_\alpha : U(\mathfrak{g}) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^\mathfrak{k}} V_\tau \rightarrow \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^\mathfrak{k}} V_\tau \\ \oplus \mathfrak{n}^\alpha U(\mathfrak{n}^\alpha) \text{symm}(\mathcal{H}_\alpha) \text{symm}(\mathcal{J}_\alpha) \otimes_{U(\mathfrak{k})U(\mathfrak{g})^\mathfrak{k}} V_\tau$$

be the projection onto the first summand. Denote by Q the projection onto the first summand of $U(\mathfrak{g}) = U(\mathfrak{a})U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g})$ followed by the projection onto $U(\mathfrak{a}) \otimes (U(\mathfrak{k})/I_\tau) = U(\mathfrak{a}) \otimes \text{End}(V_\tau)$.

Let $\rho = \frac{1}{2}\sum_{\phi \in \Phi^+} \phi$. Let α be a positive root of $\mathfrak{g}_\mathbb{R}$. Let V_ξ be an irreducible K module that occurs in $\mathcal{H} \otimes V_\tau$ and let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j^\alpha}^\alpha$ be a decomposition into K_α submodules where each $V_{\tau_j^\alpha}^\alpha$ is an irreducible K_α module such that $V_{\tau_j^\alpha}^\alpha \cong V_\tau$ as 0M modules for $j = 1, \dots, n(\xi)$. Let $\epsilon_1, \dots, \epsilon_{n(\xi)}$ be a basis of $\text{Hom}_K(V_\xi, \text{symm}(\mathcal{H}) \otimes V_\tau) \cong \text{Hom}_K(V_\xi, I_{P, \sigma, \nu})$. Let $p_{\tau_j^\alpha}$ be the determinant of $P^{\tau_j^\alpha}$ matrix of the rank one subgroup G_α with K_α type τ_j^α .

Theorem 5.7. *If α is a simple root of $\mathfrak{g}_\mathbb{R}$, $p_{\tau_j^\alpha} \mid p_\xi$.*

Proof. This proof follows that of Proposition 2.4.3 of [Kos] closely.

$(P^\xi)_{ij}$ is the action of $\epsilon_i(V_{\tau_j^\alpha}^\alpha)(e)$ followed by the replacement of elements in $\mathbb{C}[\nu]$ with the corresponding elements in $S(\mathfrak{a})$ where $e \in G$ is the identity element. Recall that $V_{\tau_j^\alpha}^\alpha$ as a K_α module is isomorphic to a submodule of $\mathcal{H}_\alpha \otimes V_\tau$ as $\mathcal{H}_\alpha \otimes V_\tau \cong \text{Ind}_{0M}^{K_\alpha}(V_\tau)$ as K_α modules (cf. 11.3.6 [RRG II]). L_α is a K_α map as $[\mathfrak{g}_\alpha, \mathfrak{n}^\alpha] \subseteq \mathfrak{n}^\alpha$ for α simple. Therefore, $(P^\xi)_{ij}$ is achieved via $L_\alpha(\epsilon_i(V_{\tau_j^\alpha}^\alpha))(e)$ as the subspace \mathfrak{n}^α acts trivially at the identity e .

Recall the definition of $V_{\tau, \alpha}^\pm$ from 4.1. Without loss of generality, assume $L_\alpha(\epsilon_i(V_{\tau_j^\alpha}^+)) = \overline{Z}_\alpha^{l_j} R_{i,j}^\alpha \otimes V_{\tau, \alpha}^+$ where $V_{\tau_j^\alpha}^+$ is the subspace of $V_{\tau_j}^\alpha$ that consist of dominant t_α weight vectors by Theorem 4.6 with $R_{i,j}^\alpha \in \text{symm}(\mathcal{J}_\alpha)$. $\text{symm}(\mathcal{J}_\alpha) \subseteq U(\mathfrak{g}_\alpha)^{\mathfrak{k}_\alpha}$ with $U(\mathfrak{g}_\alpha)^{\mathfrak{k}_\alpha}$ the subalgebra generated by t_α , the center of \mathfrak{g}_α , and the Casimir element. The action of $\overline{Z}_\alpha^{l_j} R_{i,j}^\alpha$ on $V_{\tau, \alpha}^+$ at e is $Q(\overline{Z}_\alpha^{l_j} R_{i,j}^\alpha)$. By 3.5.6 of [RRG I], $Q(\overline{Z}_\alpha^{l_j} R_{i,j}^\alpha) = Q(R_{i,j}^\alpha)Q(\overline{Z}_\alpha^{l_j}) = r_{i,j}^\alpha Q(\overline{Z}_\alpha^{l_j})$ where the action of $r_{i,j}^\alpha$ on $V_{\tau, \alpha}^+$ is invariant under $\tilde{x}_\alpha = T_{-\rho} x_\alpha T_\rho$ the translated Weyl group element of simple reflection. $Q(\overline{Z}_\alpha^{l_j})$ is an element of $U(\mathfrak{a}) \otimes (U(\mathfrak{k})/I_\tau) = U(\mathfrak{a}) \otimes \text{End}(V_\tau) = U(\mathfrak{a}) \otimes (U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M]) / (\ker \tau \cap (U(\mathfrak{k}_\alpha)\Pi\mathbb{C}[{}^0M]))$.

Hence the action of $Q(\overline{Z}_\alpha^{l_j})$ on $V_{\tau_j, \alpha}^+$ is the determinant $p_{\tau_j}^\alpha$ of P^{τ_j} matrix for the rank one subgroup G_α with K_α type τ_j , and $p_{\tau_j}^\alpha$ divides p_ξ . \square

Recall the notation of 4.1. Let ρ_ϕ play the role of ρ for the rank one subalgebra \mathfrak{g}_ϕ for a positive root ϕ of $\mathfrak{g}_\mathbb{R}$. Define $p_\phi = p_{\tau_1}^\phi \dots p_{\tau_{n(\xi)}}^\phi$ where $p_{\tau_j}^\phi$ denotes the determinant of P^{τ_j} matrix of the rank one subgroup G_ϕ with K_ϕ type τ_j . Define $p_{(\phi)} = T_{\rho_\phi - \rho} p_\phi$ where $T_{\rho_\phi - \rho}$ is the translation by $\rho_\phi - \rho$. Each $p_{\tau_j}^\phi$ is a polynomial in $h_\phi \in \mathfrak{a}$. Hence p_ϕ and $p_{(\phi)}$ are also. As $T_{\rho_\phi - \rho}(h_\phi) = h_\phi$ for ϕ simple, $p_\phi = p_{(\phi)}$ for ϕ simple.

Theorem 5.8. $p_{(\phi)} \mid p_\xi$ for any $\phi \in \Phi^+$.

Proof. This proof is almost the same as that of Proposition 2.4.5 of [Kos].

Denote by $\Delta \subset \Phi^+$ the simple root system. Let $\phi \in \Phi^+$ and let $O(\phi) = \sum m_i$ if $\phi = \sum_{\alpha_i \in \Delta} m_i \alpha_i$. If $O(\phi) = 1$, $\phi \in \Delta$ hence the claim is true by Theorem 5.7. We proceed by induction on $O(\phi)$. Assume $O(\phi) > 1$ and assume the claim for all $\psi \in \Phi^+$ with $O(\psi) < O(\phi)$. For some $\alpha \in \Delta$, $\langle \phi, h_\alpha \rangle > 0$ and let $\psi \in \Phi^+$ be such that $O(\psi) < O(\phi)$. $\phi = x_\alpha \cdot \psi$ for some $\alpha \in \Delta$ where x_α is the Weyl group element of simple reflection. Note $\psi \neq \alpha$. $p_\xi = r_\alpha p_\alpha$ where r_α is invariant under the action of \tilde{x}_α (cf. proof of Theorem 5.7). By induction hypothesis, $p_{(\psi)}$ divides $r_\alpha p_\alpha$. As p_α is a polynomial in h_α whereas $p_{(\psi)}$ is a polynomial in h_ψ , and as $h_\alpha \neq h_\psi$, p_α and $p_{(\psi)}$ are mutually prime. Hence $p_{(\psi)}$ divides r_α , therefore $\tilde{x}_\alpha \cdot p_{(\psi)}$ does also.

We assert $\tilde{x}_\alpha \cdot p_{(\psi)} = p_{(\phi)}$ up to a nonzero scalar. As $x_\alpha \cdot \psi = \phi$, $x_\alpha \cdot \mathfrak{g}_\psi = \mathfrak{g}_\phi$, $x_\alpha \cdot \mathfrak{k}_\psi = \mathfrak{k}_\phi$, and $x_\alpha \cdot \mathfrak{p}_\psi = \mathfrak{p}_\phi$. Moreover, $x_\alpha \cdot \mathfrak{a} = \mathfrak{a}$ and $x_\alpha \cdot \mathfrak{n}_\psi = \mathfrak{n}_\phi$. Therefore, for $u \in U(\mathfrak{g}_\psi)$, $x_\alpha \cdot Q(u) = Q(x_\alpha \cdot u)$. Also, $x_\alpha K_\psi x_\alpha^{-1} = K_\phi$. Furthermore, let $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}^\psi$ be a decomposition into K_ψ submodules such that each $V_{\tau_j}^\psi$ is an irreducible K_ψ module isomorphic to $V_\tau|_{\mathfrak{o}_M}$ when restricted to \mathfrak{o}_M . If $V_{\tau_j}^\phi = x_\alpha V_{\tau_j}^\psi$, then $V_\xi = \bigoplus_{j=1}^{n(\xi)} V_{\tau_j}^\phi$ is a decomposition into K_ϕ submodules such that each $V_{\tau_j}^\phi$ is an irreducible K_ϕ module isomorphic to $V_\tau|_{\mathfrak{o}_M}$ when restricted to \mathfrak{o}_M by Lemma 4.11. Hence $x_\alpha \cdot p_\psi = p_\phi$ up to a nonzero scalar. But, $\tilde{x}_\alpha \cdot p_{(\psi)} = T_{-\rho} x_\alpha T_\rho T_{\rho_\psi - \rho} p_\psi = T_{-\rho} x_\alpha T_{\rho_\psi} p_\psi = T_{-\rho} x_\alpha T_{\rho_\psi} x_\alpha^{-1} x_\alpha \cdot p_\psi = T_{-\rho} T_{x_\alpha \cdot \rho_\psi} x_\alpha \cdot p_\psi = T_{\rho_\phi - \rho} p_\phi = p_{(\phi)}$, and this completes the assertion and $p_{(\phi)}$ divides p_ξ . \square

5.3. Let V_ξ be an irreducible K module that occurs in $\mathcal{H} \otimes V_\tau$. Recall the notation of section 3 and section 5.2. We are now ready to prove Theorem 2 in the introduction.

Proof. (of Theorem 2)

The right hand side divides the left hand side by Theorem 5.8 and the fact that for $\phi, \psi \in \Phi^+$ with $\phi \neq \psi$, $p_{(\phi)}$ and $p_{(\psi)}$ are mutually prime. We assert the degrees of the two polynomials are equal. As the definition of p_ξ is independent of the basis up to a nonzero scalar, we may assume $e_i^\xi(T_j^\xi(V_\tau)) \subseteq \text{symm}(\mathcal{H})_{\deg(i)} \otimes V_\tau$, i.e. a homogeneous basis. Therefore, the degree of the left hand side, $\deg(\xi)$, is at most $\sum_{i=1}^{n(\xi)} \deg(i)$. If $\gamma_1, \dots, \gamma_N$ are the distinct K types that occur in $\text{symm}(\mathcal{H})$ with $V_\xi \subseteq V_{\gamma_j} \otimes V_\tau$, $\sum_{i=1}^{n(\xi)} \deg(i) = \sum_{j=1}^N \deg(\gamma_j)$ where $\deg(\gamma_j)$ is the sum of the degrees in which V_{γ_j} occur in $\text{symm}(\mathcal{H})$. But, $\deg(\gamma_j) = \sum_{\phi \in \Phi^+} n_{\gamma_j}^\phi$ where $n_{\gamma_j}^\phi$ is the sum of the degrees in which the irreducible K_ϕ submodules in $\text{Span } K_\phi \cdot V_{\gamma_j}^{0M}$ occur in $\text{symm}(\mathcal{H}_\phi)$ by Proposition 2.3.12 and Theorem 2.3.14 of [Kos]. $n_{\gamma_j}^\phi$ only depends on the unique dominant t_ϕ weights (cf. Theorem 4.5). Hence $\sum_{i=1}^{n(\xi)} \deg(i) = \sum_{j=1}^N \deg(\gamma_j) = \sum_{j=1}^N \sum_{\phi \in \Phi^+} n_{\gamma_j}^\phi = \sum_{\phi \in \Phi^+} \deg(p_{(\phi)}(\nu))$ by Theorem 5.4 and the fact that $\deg(Q(Z_\phi^l)) = l$ by Theorem 7.6 of [JW]. Therefore the degree of the left hand side is less than or equal to the degree of the right hand side. As the right hand side divides the left hand side, the degree of the right hand side is less than or equal to the degree of the left hand side. Hence the degrees of the two polynomials are equal. \square

6. APPLICATIONS

In this section, we state some applications of the product formula for p_ξ . Let $\mathfrak{g}_\mathbb{R}$ be any of the split real form of simply laced, simple Lie type of rank ≥ 2 . Let G be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{R}$. In 6.1, the intertwining operator between the genuine principal series representations of G is realized as a ratio of the P^ξ matrix. In 6.2, we compute p_ξ explicitly for the rank one case of type A_1 . As a result, we derive a formula of p_ξ and the determinant of the intertwining operator between the genuine principal series representations of $SL(n, \mathbb{R})$. In 6.3, we classify irreducible (\mathfrak{g}, K) modules that admit a small K type τ . We note that in this section, the principal series representation, the intertwining operator between the principal series representations, and $P^\xi(\nu)$ are with ρ shifts.

6.1. Let (ρ_ξ, V_ξ) be an irreducible K module that occurs in $\mathcal{H} \otimes V_\tau$ for a small K type τ . Let $\sigma = V_\tau|_{\mathfrak{o}_M}$. Recall the notation of section 3. $P^\xi(\nu)$

is a map of $\bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau) \longrightarrow \bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau)$ by setting $P^\xi(\nu)T_i^\xi(v) = \sum_{j=1}^{n(\xi)} T_j^\xi(P_{ji}^\xi(\nu)v)$ for $v \in V_\tau$. Let $V_\xi = \bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau)$ be a decomposition of V_ξ as a 0M module. We define a map of $P^\xi(\nu)$ on V_ξ where $P^\xi(\nu)$ acts as above on $\bigoplus_{j=1}^{n(\xi)} T_j^\xi(V_\tau)$. Thus, we may consider $P^\xi(\nu)$ as an operator on $\text{Hom}_{{}^0M}(V_\xi, V_\tau)$ where $P^\xi(\nu) \cdot \lambda = \lambda \circ P^\xi(\nu)$ for $\lambda \in \text{Hom}_{{}^0M}(V_\xi, V_\tau)$.

For $\lambda \in \text{Hom}_{{}^0M}(V_\xi, V_\tau)$ and $v \in V_\xi$, define $(\lambda \otimes v)(k) = \lambda(\rho_\xi(k)v)$ that gives a K module isomorphism $\text{Hom}_{{}^0M}(V_\xi, V_\tau) \otimes V_\xi \cong I_{P,\sigma,\nu}(\xi)$ where $I_{P,\sigma,\nu}(\xi)$ is the ξ isotypic component of $I_{P,\sigma,\nu}$. Define for $a \in \text{Hom}_K(V_\xi, \text{symm}(\mathcal{H}) \otimes V_\tau)$ and $v \in V_\xi$, $B_\nu^\xi(a)(v) = \pi_{P,\sigma,\nu}(a(v))(e)$. $B_\nu^\xi : \text{Hom}_K(V_\xi, \text{symm}(\mathcal{H}) \otimes V_\tau) \longrightarrow \text{Hom}_{{}^0M}(V_\xi, V_\tau)$. Let $T_\nu : \text{symm}(\mathcal{H}) \otimes V_\tau \longrightarrow I_{P,\sigma,\nu}$ be defined by $T_\nu(\Sigma(u_j \otimes v_j)) = \Sigma(\pi_{P,\sigma,\nu}(u_j)v_j)$. Let $\nu_0 \in \mathfrak{a}^*$ be such that T_{ν_0} is a bijection (11.3.6 [RRG II]).

Lemma 6.1. $T_\nu \circ T_{\nu_0}^{-1}(\lambda \otimes v) = \lambda \circ P^\xi(\nu) \otimes v$ for $\lambda \in \text{Hom}_{{}^0M}(V_\xi, V_\tau)$ and $v \in V_\xi$.

Proof. This proof is almost the same as that of Lemma 7.3 of [JW].

Let $\hat{\delta}_\xi : \text{Hom}_K(V_\xi, \text{symm}(\mathcal{H}) \otimes V_\tau) \longrightarrow \text{Hom}_{{}^0M}(V_\xi, V_\tau)$ be defined so that $T_{\nu_0}(a(v)) = \hat{\delta}_\xi(a) \otimes v$ for $a \in \text{Hom}_K(V_\xi, \text{symm}(\mathcal{H}) \otimes V_\tau)$ and $v \in V_\xi$. By the above discussion, $B_{\nu_0}^\xi(a) = \hat{\delta}_\xi(a)$. $T_\nu \circ T_{\nu_0}^{-1}(B_{\nu_0}^\xi(a) \otimes v) = B_\nu^\xi(a) \otimes v$. But $B_\nu^\xi(a_i)(T_j^\xi(V_\tau)) = P_{ij}^\xi(\nu)$ where the basis $\{a_i\}$ of $\text{Hom}_{{}^0M}(V_\xi, V_\tau)$ and $\{T_j^\xi(V_\tau)\}$ can be chosen so that $P^\xi(\nu_0)$ is an identity matrix as $P^\xi(\nu_0)$ is invertible. Thus $B_\nu^\xi(a) = B_{\nu_0}^\xi(a) \circ P^\xi(\nu)$. Therefore, if $B_{\nu_0}^\xi(a) = \lambda$, $T_\nu \circ T_{\nu_0}^{-1}(\lambda \otimes v) = \lambda \circ P^\xi(\nu) \otimes v$. \square

Let \overline{P} be the opposite parabolic subgroup of a minimal parabolic subgroup P . Define $\overline{T}_\nu : \text{symm}(\mathcal{H}) \otimes V_\tau \longrightarrow I_{\overline{P},\sigma,\nu}$ similarly as above.

Theorem 6.2. Let $A(\nu) : I_{P,\sigma,\nu} \longrightarrow I_{\overline{P},\sigma,\nu}$ be such that $A(\nu)w = w$ for $w \in V_\tau$ and $A(\nu) \circ \pi_{P,\sigma,\nu}(u) = \pi_{\overline{P},\sigma,\nu}(u) \circ A(\nu)$ for all $u \in U(\mathfrak{g})$. Then

$$A(\nu)(\lambda \otimes v) = \lambda \circ P^\xi(\nu)^{-1} P^\xi(-\nu) \otimes v$$

for $\lambda \in \text{Hom}_{{}^0M}(V_\xi, V_\tau)$ and $v \in V_\xi$, if $\det P^\xi(\nu) \neq 0$ and $\det P^\xi(-\nu) \neq 0$ for all $\xi \in \hat{K}$ that occurs in $I_{P,\sigma,\nu}$.

Proof. This proof is almost the same as that of Lemma 7.5 of [JW].

If $u \otimes w \in \text{symm}(\mathcal{H}) \otimes V_\tau$ is a simple tensor,

$$\begin{aligned} A(\nu)T_\nu(u \otimes w) &= A(\nu)\pi_{P,\sigma,\nu}(u)w \\ &= \pi_{\overline{P},\sigma,\nu}(u)A(\nu)w \\ &= \pi_{\overline{P},\sigma,\nu}(u)w \\ &= \overline{T}_\nu(u \otimes w) \end{aligned}$$

Hence $A(\nu)(T_\nu \circ T_{\nu_0}^{-1})(\lambda \otimes v) = (\overline{T}_\nu \circ \overline{T}_{-\nu_0}^{-1})(\lambda \otimes v)$. Thus,

$$A(\nu)(\lambda \circ P^\xi(\nu) \otimes v) = \lambda \circ P^\xi(-\nu) \otimes v$$

by Lemma 6.1. Therefore,

$$A(\nu)(\lambda \otimes v) = \lambda \circ P^\xi(\nu)^{-1} P^\xi(-\nu) \otimes v$$

□

6.2. Let G be any of the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 . Recall the notation of 4.1. For a positive root α of $\mathfrak{g}_\mathbb{R}$, $\mathfrak{g}_{\mathbb{R}\alpha} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus Z(\mathfrak{g}_{\mathbb{R}\alpha})$. Recall for G_α ,

$$\mathcal{H}_\alpha = \bigoplus_{l \geq 0} Z_\alpha^l \oplus \bigoplus_{l > 0} \overline{Z}_\alpha^l$$

where $Z_\alpha = h_\alpha + iy_\alpha$. We drop the subscript α for convenience. Recall the projection map $Q : U(\mathfrak{g}) \rightarrow U(\mathfrak{a})U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g})$ onto the first summand. To compute $p_\xi(\nu)$ for a K type ξ that occurs in $\mathcal{H} \otimes V_\tau$, we compute $Q(Z^l)$ and $Q(\overline{Z}^l)$ with t weight $\pm \frac{1}{2}$. To do this, we use $Q'(Z^l)$ and $Q'(\overline{Z}^l)$ already computed in [JW] where $Q' : U(\mathfrak{g}) \rightarrow U(\mathfrak{a}) \oplus \mathfrak{n}U(\mathfrak{g}) \oplus U(\mathfrak{g})\mathfrak{k}$ is the projection onto the first summand. This is because $Q(Z^l)$ and $Q(\overline{Z}^l)$ can be written as a sum of two different parts, one in $U(\mathfrak{a})$, and the other in $U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k}$, where the former is exactly $Q'(Z^l)$ and $Q'(\overline{Z}^l)$ respectively.

Theorem 6.3. $Q(Z^l) = \Pi_{j=0}^{l-1}(h+2j-t)$ and $Q(\overline{Z}^l) = \Pi_{j=0}^{l-1}(h+2j+t)$.

Proof. We prove the first formula.

By Theorem 7.6 of [JW], $Q'(Z^l) = \Pi_{j=0}^{l-1}(h+2j)$. We find the shift from $U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k}$ part as it is the only difference between Q and Q' . If $l = 1$, then $Z = h + iy = h + i(2e + it)$, hence the formula is true. We proceed by induction. Assume the formula for $l-1$. We prove the formula for l . $Z^l = ZZ^{l-1} = (h + 2ie - t)Z^{l-1}$. After dropping the \mathfrak{n} part e , $(h-t)Z^{l-1}$ is left. There are exactly two $U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k}$ shifts, one from $h(U(\mathfrak{a})U(\mathfrak{k})\mathfrak{k})$ part of Z^{l-1} and the other from $-t(Z^{l-1}) = -Z^{l-1}(t - 2(l-1))$ by the commutation relation. Hence, the overall shift is $(h + 2(l-1))(Q(Z^{l-1}) - Q'(Z^{l-1})) - tQ(Z^{l-1})$. As $Q'(Z^l) =$

$$(h + 2(l - 1))Q'(Z^{l-1}),$$

$$\begin{aligned} Q(Z^l) &= Q'(Z^l) + \text{shift} \\ &= (h + 2(l - 1))Q'(Z^{l-1}) \\ &\quad + (h + 2(l - 1))(Q(Z^{l-1}) - Q'(Z^{l-1})) - tQ(Z^{l-1}) \\ &= (h + 2(l - 1))Q(Z^{l-1}) - tQ(Z^{l-1}) \\ &= (h + 2(l - 1) - t)Q(Z^{l-1}) \end{aligned}$$

Hence the first formula is proved. The second formula is proved similarly. \square

For the ξ type $\overline{Z}^l \otimes V_\tau^+ \oplus Z^l \otimes V_\tau^-$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j + \frac{3}{2})$$

and for the ξ type $\overline{Z}^l \otimes V_\tau^- \oplus Z^l \otimes V_\tau^+$,

$$p_\xi(\nu) = \Pi_{j=0}^{l-1}(\nu + 2j + \frac{1}{2})$$

Consider the group $\widetilde{SL(n, \mathbb{R})}$. Define $q_\nu : 2\mathbb{N} + 1 \rightarrow \mathbb{C}[\nu]$ as follows.

$$\begin{aligned} q_\nu(m) &:= \Pi_{l=0}^{\frac{m-1}{4}} \Pi_{j=0}^{l-1}(\nu + 2j + \frac{1}{2})(\nu + 2j + \frac{3}{2}) \text{ if } 4 \mid m - 1 \\ q_\nu(m) &:= \Pi_{l=0}^{\frac{m-3}{4}} \Pi_{j=0}^{l-1}(\nu + 2j + \frac{1}{2})(\nu + 2j + \frac{3}{2}) \\ &\quad \times \Pi_{k=0}^{\frac{m-3}{4}}(\nu + 2k + \frac{1}{2}) \text{ if } 4 \mid m - 3 \end{aligned}$$

Define $G_\nu(m) : 2\mathbb{N} + 1 \rightarrow M$ where M is the space of meromorphic functions, as follows.

$$\begin{aligned} \Gamma_\nu(m) &:= \Pi_{l=0}^{\frac{m-1}{4}} \Pi_{j=0}^{l-1} \frac{\Gamma(\nu - 2j + \frac{1}{2})}{\Gamma(\nu - 2j - \frac{3}{2})} \frac{\Gamma(\nu + 2j + \frac{1}{2})}{\Gamma(\nu + 2j + \frac{5}{2})} \text{ if } 4 \mid m - 1 \\ \Gamma_\nu(m) &:= \Pi_{l=0}^{\frac{m-3}{4}} \Pi_{j=0}^{l-1} \frac{\Gamma(\nu - 2j + \frac{1}{2})}{\Gamma(\nu - 2j - \frac{3}{2})} \frac{\Gamma(\nu + 2j + \frac{1}{2})}{\Gamma(\nu + 2j + \frac{5}{2})} \times \Pi_{k=0}^{\frac{m-3}{4}} \frac{\Gamma(\nu - 2k + \frac{1}{2})}{\Gamma(\nu - 2k - \frac{3}{2})} \text{ if } 4 \mid m - 3 \end{aligned}$$

Let V_ξ be an irreducible $K = Spin(n)$ module that occurs in $I_{P, \sigma, \nu}$. The dominant t_α weights of interest on V_ξ counting multiplicity are independent of $\alpha \in \Phi^+$. Hence, we find the dominant t_α weights of interest on V_ξ for $\alpha = \epsilon_1 - \epsilon_2$. We branch down V_ξ to $Spin(3)$ instead of $Spin(2) \cong SO(2)$ to simplify notation, where $Spin(3)$ is such that the quotient group $SO(3)$ occurs in the top left corner of the quotient group $SO(n)$ of $Spin(n)$. Let $n = 2k + 1$ or $n = 2k$ and let $\xi = \xi_1 \epsilon_1 + \dots + \xi_k \epsilon_k$ be the highest weight of ξ . Let $\frac{j_1}{2}, \dots, \frac{j_{m_\xi}}{2}$ be the highest weights of irreducible $Spin(3)$ modules that occur in the branching counting multiplicity.

$$p_\xi(\nu) = (\Pi_{\alpha \in \Phi^+} \Pi_{k=1}^{m_\xi} q_{(\nu, \alpha)}(j_k))^{\frac{2}{\dim(V_\tau)}}$$

$$\begin{aligned} \det A(\nu)|_{I_{P, \tau, \nu}(\xi)} &= \left(\frac{p_\xi(-\nu)}{p_\xi(\nu)} \right)^{\dim(V_\xi)} \\ &= ((\Pi_{\alpha \in \Phi^+} \Pi_{k=1}^{m_\xi} \Gamma_{(\nu, \alpha)}(j_k))^{\frac{2}{\dim(V_\tau)}})^{\dim(V_\xi)} \end{aligned}$$

where if $n = 2k + 1$, $\dim(V_\xi) = \prod_{1 \leq i < j \leq k} \frac{(\xi_i + \rho_i)^2 - (\xi_j + \rho_j)^2}{\rho_i^2 - \rho_j^2} \prod_{1 \leq i \leq k} \frac{\xi_i + \rho_i}{\rho_i}$ with $\rho_i = k - i + \frac{1}{2}$ (cf. 7.1.2 [GW]), $\dim(V_\tau) = 2^k$, and if $n = 2k$, $\dim(V_\xi) = \prod_{1 \leq i < j \leq k} \frac{(\xi_i + \rho_i)^2 - (\xi_j + \rho_j)^2}{\rho_i^2 - \rho_j^2}$ with $\rho_i = k - i$ (cf. 7.1.2 [GW]), $\dim(V_\tau) = 2^{k-1}$.

6.3. Consider the (\mathfrak{g}, K) module homomorphism

$$\mu_{\tau, \nu} : U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^* U(\mathfrak{k})} V_{\tau, \nu} \longrightarrow I_{P, \sigma, \nu}$$

from 11.3.6 of [RRG II] where the homomorphism is the action of the first factor on the second as differential operators. $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^* U(\mathfrak{k})} V_{\tau, \nu} \cong I_{P, \sigma, \nu}$ as K modules (11.3.6 [RRG II]). Therefore, the small K type $I_{P, \sigma, \nu}(\tau)$ is cyclic if and only if $\mu_{\tau, \nu}$ is a (\mathfrak{g}, K) module isomorphism. By the definition of $P^\xi(\nu)$, $I_{P, \sigma, \nu}(\tau)$ is cyclic if and only if $p_\xi(\nu) \neq 0$ for every $\xi \in \hat{K}$ that occurs in $I_{P, \sigma, \nu}$.

Theorem 6.4. *Let G be the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 . If $\operatorname{Re}(\nu, \alpha) \geq 0$ for every $\alpha \in \Phi^+$, i.e. in the closed Langlands chamber, $I_{P, \sigma, \nu}(\tau)$ is cyclic.*

Proof. By Theorem 2, $p_\xi(\nu)$ up to a nonzero scalar is the product of those of rank one subgroups G_α of G where $\alpha \in \Phi^+$. As G is split, the semisimple part of $\mathfrak{g}_{\mathbb{R}_\alpha}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Let $n(\xi) = \dim \operatorname{Hom}_K(V_\tau, V_\xi)$. The product formula for p_ξ and the formulas in the rank one case imply that for a K type ξ that occurs in $I_{P, \sigma, \nu}$, $\alpha \in \Phi^+$, and $j = 1, \dots, n(\xi)$, there exist $l_{\alpha, j}^\xi, m_{\alpha, j}^\xi \in \mathbb{Z}_{\geq 0}$ such that $p_\xi(\nu)$ is equal to

$$\Pi_{\alpha \in \Phi^+} \Pi_{j=1}^{n(\xi)} \Pi_{p=0}^{l_{\alpha, j}^\xi - 1} \Pi_{q=0}^{m_{\alpha, j}^\xi - 1} \left(\frac{2(\nu, \alpha)}{(\alpha, \alpha)} + 2p + \frac{1}{2} \right) \left(\frac{2(\nu, \alpha)}{(\alpha, \alpha)} + 2q + \frac{3}{2} \right)$$

up to a nonzero scalar multiple. Therefore, if $\operatorname{Re}(\nu, \alpha) \geq 0$ for every $\alpha \in \Phi^+$, $p_\xi(\nu) \neq 0$ for every K type ξ that occurs in $I_{P, \sigma, \nu}$ and $I_{P, \sigma, \nu}(\tau)$ is cyclic. \square

Theorem 6.5. *Let G be the connected, simply connected split real form of simply laced, simple Lie type of rank ≥ 2 . The underlying (\mathfrak{g}, K) module $I_{P,\sigma,\nu}$ ($\operatorname{Re} \nu = 0$) of the unitary principal series is irreducible.*

Proof. If $I_{P,\sigma,\nu}$ with $\operatorname{Re} \nu = 0$ is reducible, there is a proper, nontrivial, closed (\mathfrak{g}, K) invariant subspace W of $I_{P,\sigma,\nu}$ that does not contain τ as $I_{P,\sigma,\nu}(\tau)$ is cyclic by Theorem 6.4. Unitarity implies that the orthogonal complement of W , W^\perp , is a proper, nontrivial, closed (\mathfrak{g}, K) invariant subspace that contains τ , which is a contradiction as $I_{P,\sigma,\nu}(\tau)$ is cyclic. \square

We assert that there is a bijection between the set of equivalence classes of irreducible (\mathfrak{g}, K) modules admitting a small K type τ and the set of Weyl group orbits in \mathfrak{a}^* as in the spherical case. Indeed, let V be an irreducible (\mathfrak{g}, K) module admitting a small K type τ . As $V_\tau|_{\mathfrak{o}_M}$ is irreducible, V must be (\mathfrak{g}, K) isomorphic to a subquotient of $I_{P,\sigma,\nu}$ for some $\nu \in \mathfrak{a}^*$ by the Harish-Chandra Subquotient Theorem (cf. Theorem 3.5.6 [RRG I]). Therefore, V is completely determined by the action of $U(\mathfrak{g})^K$ on $V(\tau)$ by Proposition 3.5.4 of [RRG I]. Let Q be the projection onto the first summand of $U(\mathfrak{g}) = U(\mathfrak{a})U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g})$ and let $\varpi(X) = X + \rho(X)$ for $X \in \mathfrak{a}_{\mathbb{R}}^*$. The action of $U(\mathfrak{g})^K$ on $I_{P,\sigma,\nu}(\tau)$ is given by $((\nu \circ \varpi) \otimes \tau)(Q(g))$ for $g \in U(\mathfrak{g})^K$. There exists a map $\gamma_\tau : U(\mathfrak{g})^K \rightarrow U(\mathfrak{a})^{N_K(A)/Z_K(A)}$ such that $(\varpi \otimes \tau)(Q(g)) = \gamma_\tau(g)$ (cf. Lemma 11.3.2 of [RRG II]). In fact, γ_τ gives an algebra isomorphism $U(\mathfrak{g})^K / (U(\mathfrak{g})^K \cap U(\mathfrak{g})J_\tau) \cong U(\mathfrak{a})^{N_K(A)/Z_K(A)}$ where $J_\tau = \ker \tau \subset U(\mathfrak{k})$. If $\nu, \nu' \in \mathfrak{a}^*$ and $\nu(f) = \nu'(f)$ for all $f \in U(\mathfrak{a})^{N_K(A)/Z_K(A)}$, there exists $s \in N_K(A)/Z_K(A)$ such that $s\nu = \nu'$ (cf. proof of Theorem 3.1.2 [RRG I]). We are now ready to prove Theorem 3 in the introduction.

Proof. (of Theorem 3)

By the above, we may assume V is (\mathfrak{g}, K) isomorphic to a subquotient of $I_{P,\sigma,\nu}$ for some $\nu \in \mathfrak{a}^*$ in the closed Langlands chamber. By Theorem 6.4, $\mu_{\tau,\nu} : Y^{\tau,\nu} \rightarrow I_{P,\sigma,\nu}$ is a (\mathfrak{g}, K) module isomorphism in the closed Langlands chamber. As $Y^{\tau,\nu}$ has a unique irreducible quotient (cf. 3.5.4 [RRG I]), the Theorem is proved. \square

Let V be an irreducible (\mathfrak{g}, K) module that admits a small K type τ . By Theorem 3, V is (\mathfrak{g}, K) isomorphic with the unique irreducible quotient of $I_{P,\sigma,\nu}$ for some $\nu \in \mathfrak{a}^*$ in the closed Langlands chamber. Suppose $\operatorname{Re}(\nu, \alpha) > 0$ for some $\alpha \in \Phi^+$. Let $F = \{\alpha \in \Delta \mid \operatorname{Re}(\nu, \alpha) = 0\}$ and let $\mathfrak{a}_{\mathbb{R}F} = \{H \in \mathfrak{a}_{\mathbb{R}} \mid \alpha(H) = 0 \text{ for all } \alpha \in F\}$. Let ${}^0M_F = Z_G(\mathfrak{a}_{\mathbb{R}F})$ and let $P_F = {}^0M_F A_F N_F$ be a given Langlands decomposition of a parabolic subgroup P_F with special vector subgroup $A_F = \exp(\mathfrak{a}_{\mathbb{R}F})$. Let $\nu = \varsigma + \mu$ where $\varsigma(H) = 0$, $\mu(H) = \nu(H)$ for $H \in \mathfrak{a}_{\mathbb{R}F}$ and $\varsigma(H) = \nu(H)$,

$\mu(H) = 0$ for $H \in \mathfrak{a}_{\mathbb{R}} - \mathfrak{a}_{\mathbb{R}F}$. $I_{P,\sigma,\nu}$ is realized in induction steps as $I_{P_F, I_{0_{M_F \cap P, \sigma, \varsigma}}, \mu}$ where $(\sigma_F, I_{0_{M_F \cap P, \sigma, \varsigma}})$ is a tempered, unitary representation of 0M_F and μ is in the Langlands chamber. $(\sigma_F, I_{0_{M_F \cap P, \sigma, \varsigma}})$ is irreducible as Ind is an exact functor and there exists an irreducible, unitary principal series representation $I_{P, \sigma, \chi}$ that is realized in induction steps as $I_{P_F, I_{0_{M_F \cap P, \sigma, \varsigma}}, \chi'}$ for some $\chi' \in \mathfrak{a}^*$. Otherwise, $Re(\nu, \alpha) = 0$ for all $\alpha \in \Phi^+$. $I_{P, \sigma, \nu}$ then is a tempered, unitary principal series which is irreducible by Theorem 6.5.

Corollary 6.6. *Langlands parameters for V are:*

- (P_F, σ_F, μ) if $Re(\nu, \alpha) > 0$ for some $\alpha \in \Phi^+$.
- Tempered if $Re(\nu, \alpha) = 0$ for all $\alpha \in \Phi^+$.

Corollary 6.7. *V cannot be equivalent to a discrete series representation.*

Proof. Let F be as above. If $F \neq \Delta$, then V is not tempered. If $F = \Delta$, then V is equivalent with a unitary principal series representation for a proper parabolic subgroup which cannot be square integrable. \square

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